بسم الله الرحمن الرحيم

تلخيص رياضيات ٢

للدكتور كامل فليفل. || إعداد: "مهندس متشائل" || شبكة المهندس المسلم

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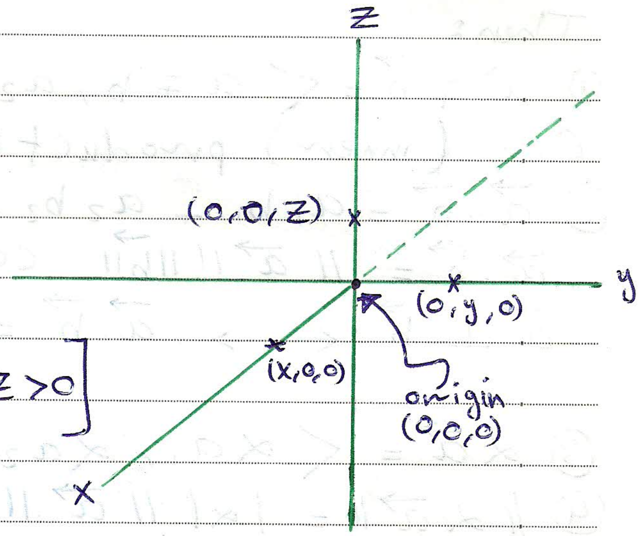
Coordinate planes:

3-D space



$x, y \rightarrow$ plane $z=0$

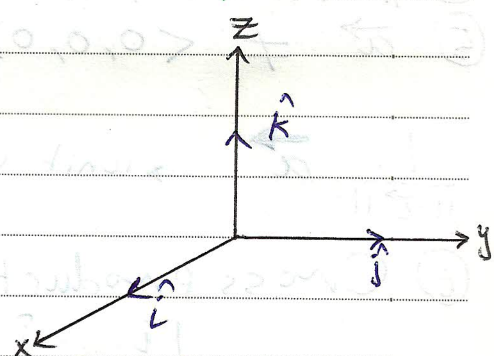
$x, z \rightarrow$ plane $y=0$

$y, z \rightarrow$ plane $x=0$

First octant $= \left[(x,y,z) : x>0, y>0, z>0 \right]$

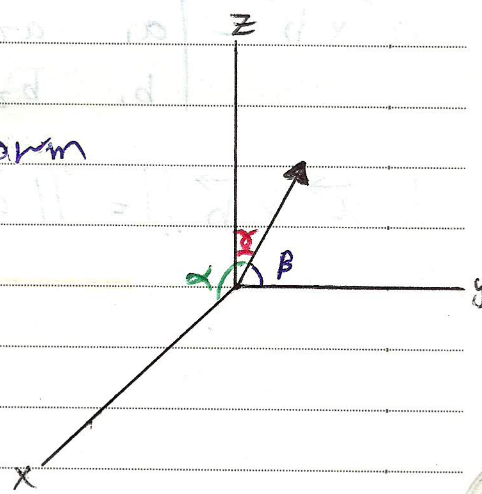
$\therefore$ any vector can be written as

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$
$$= a\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$$



$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\|\vec{a}\| = |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \text{norm}$$



$$\cos\alpha = \frac{a_1}{\|\vec{a}\|}$$

$$\cos\beta = \frac{a_2}{\|\vec{a}\|}$$

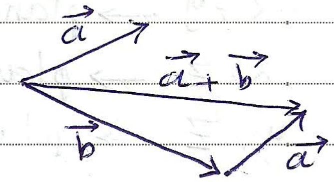
$$\cos\gamma = \frac{a_3}{\|\vec{a}\|}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

Then:

① $\vec{a} \mp \vec{b} = \langle a \mp b, a_2 \mp b_2, a_3 \mp b_3 \rangle$



Dot (inner) product:

② $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$

$\vec{a} \perp \vec{b} \longleftrightarrow \vec{a} \cdot \vec{b} = 0$ $\perp$: perpendicular (orthogonal)

③ $\alpha \vec{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$ α: scalar كمية قياسية

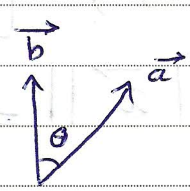
④ $\|\alpha \vec{a}\| = |\alpha| \, \|\vec{a}\|$

⑤ $\vec{a} \neq \langle 0, 0, 0 \rangle = \vec{0}$

$\frac{1}{\|\vec{a}\|} \vec{a}$ → unit vector in the direction of $\vec{a}$

⑥ Cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$

Example:

$\vec{a} = \langle -1, 2, 2 \rangle$

$\vec{b} = \langle 3, 4, 0 \rangle$

Find:

① $\vec{a} + \vec{b}$, $\|\vec{a} + \vec{b}\|$

② $\|\vec{a}\|$, $\|\vec{b}\|$

③ $\vec{a} \cdot \vec{b}$

④ $\vec{a} \times \vec{b}$

⑤ $\vec{a} \cdot (\vec{a} \times \vec{b})$

⑥ a unit vector in the direction of $\vec{a}$.

Solve:

① $\vec{a} + \vec{b} = \langle 2, 6, 2 \rangle$

$\|\vec{a} + \vec{b}\| = \sqrt{4+36+4} = \sqrt{44}$

② $\|\vec{a}\| = \sqrt{1+4+4} = 3$

$\|\vec{b}\| = \sqrt{9+16} = 5$

* $\|\vec{a} + \vec{b}\| \neq \|\vec{a}\| + \|\vec{b}\|$ Triavigular inquality

$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$ يكونان متساويان إذا كان أحدهما صفر أو الاثنان في نفس الاتجاه

③ $\vec{a} \cdot \vec{b} = (-1)(3) + (2)(4) + (2)(0) = 5$

④ $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 3 & 4 & 0 \end{vmatrix} = (0-8)\hat{i} - (0-6)\hat{j} + (-4-6)\hat{k}$

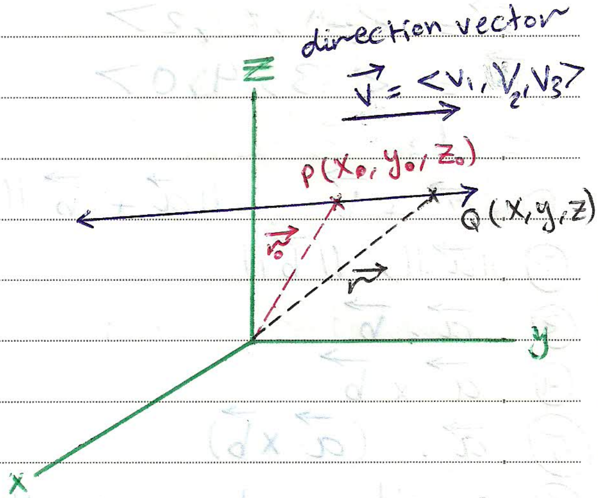
$= -8\hat{i} + 6\hat{j} - 10\hat{k}$

⑤ $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0 \longrightarrow \perp$

$\langle -1, 2, 2 \rangle \cdot \langle -8, 6, -10 \rangle = 8 + 12 - 20 = 0$

⑥ $\hat{u} = \dfrac{1}{\|\vec{a}\|} \vec{a} = \left\langle \dfrac{-1}{3}, \dfrac{2}{3}, \dfrac{2}{3} \right\rangle$

Equation of a line in the space :



$\overrightarrow{PQ} = \langle x - x_0 , y - y_0 , z - z_0 \rangle$

$= \vec{r} - \vec{r_0}$

$(\vec{r} - \vec{r_0}) \parallel \vec{V}$

$(\vec{r} - \vec{r_0}) = t \vec{V} \quad , t \in R$

$$\boxed{\vec{r} = \vec{r_0} + t \vec{V}}$$

vector equation of the line

$\langle x - x_0 , y - y_0 , z - z_0 \rangle = t \langle V_1 , V_2 , V_3 \rangle$

$x - x_0 = t V_1 \longrightarrow x = x_0 + t V_1$

$y - y_0 = t V_2 \longrightarrow y = y_0 + t V_2 \qquad t \in R$

$z - z_0 = t V_3 \longrightarrow z = z_0 + t V_3$

$$\frac{x - x_0}{V_1} = \frac{y - y_0}{V_2} = \frac{z - z_0}{V_3}$$ symmetric equations

e.g: Find the parametric of the line that :
passes through the point $(2, 4, -1)$ and
Paralled to $3\hat{i} + 5\hat{j} - 7\hat{k}$

The parametric equations:

$x = 2 + 3t$

$y = 4 + 5t \qquad t \in R$

$z = -1 - 7t$

e.g: Find the parametric equations of the line that passes through the points:
A $(1,2,3)$, B $(-1,4,0)$

$\overrightarrow{AB} = \langle -2, 2, -3 \rangle$

$x = 1 - 2t$
$y = 2 + 2t$ $\quad t \in R$
$z = 3 - 3t$

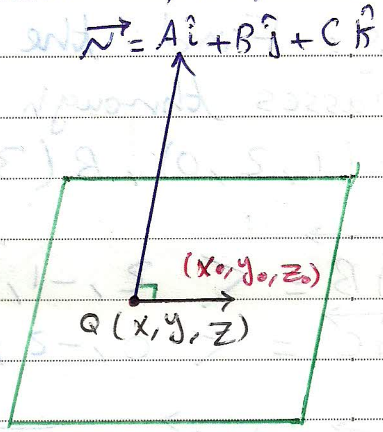
or

$x = -1 - 2M$
$y = 4 + 2M$ $\quad M \in R$
$z = -3M$

The equation of the plane:
The equation of the plane that passes through point (x_0, y_0, z_0) and normal (perpendicular) to the vector: $\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$
is:

$$\boxed{\vec{N} \cdot \overrightarrow{PQ} = 0}$$

$\langle x - x_0, y - y_0, z - z_0 \rangle$



e.g: Find the eqation of the plane that passes through the point $(1, -1, 3)$, and normal to the vector: $2\hat{i} + 4\hat{j} - 5\hat{k}$

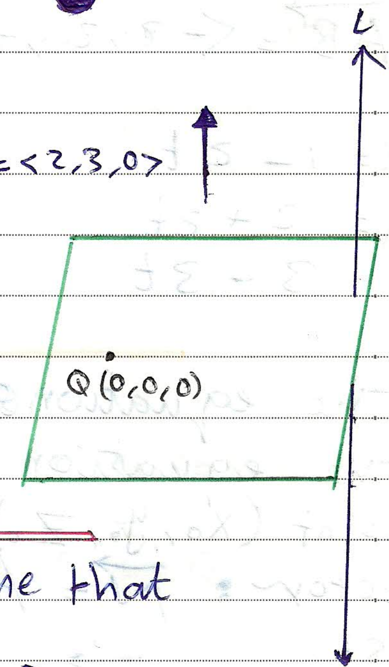
$2(x-1) + 4(y+1) - 5(z-3) = 0$
$2x - 2 + 4y + 4 - 5z - 15 = 0$
$2x + 4y - 5z + 13 = 0$

e.g: Find the equation of the plane that passes through the origin and normal to the line $L: \begin{cases} x = 1 + 2t \\ y = 3t \\ z = 5 \end{cases} \quad t \in \mathbb{R}$

$\therefore$ The equation of the plane is:

$$2(x-0) + 3(y-0) + 0(z-0) = 0$$

$$2x + 3y = 0$$



e.g: Find the equation of the plane that passes through the points:
A (1,2,0), B (3,1,2) and C (1,0,4)

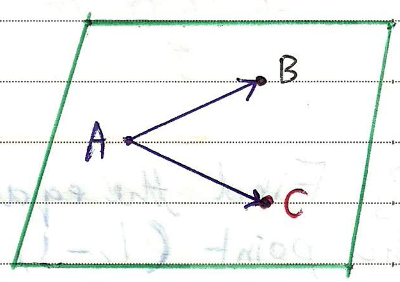
$\overrightarrow{AB} = \langle 2, -1, 2 \rangle$

$\overrightarrow{AC} = \langle 0, -2, 4 \rangle$

$$\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 0 & -2 & 4 \end{vmatrix}$$

$$\vec{N} = (0)\hat{i} - (8-0)\hat{j} + (-4-0)\hat{k}$$

$$\vec{N} = -8\hat{j} - 4\hat{k}$$



The equation of the plane is:

$$0(x-1) - 8(y-2) - 4(z-0) = 0$$

$$-8y + 16 - 4z = 0 \rightarrow 2y + z - 4 = 0$$

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

Then:

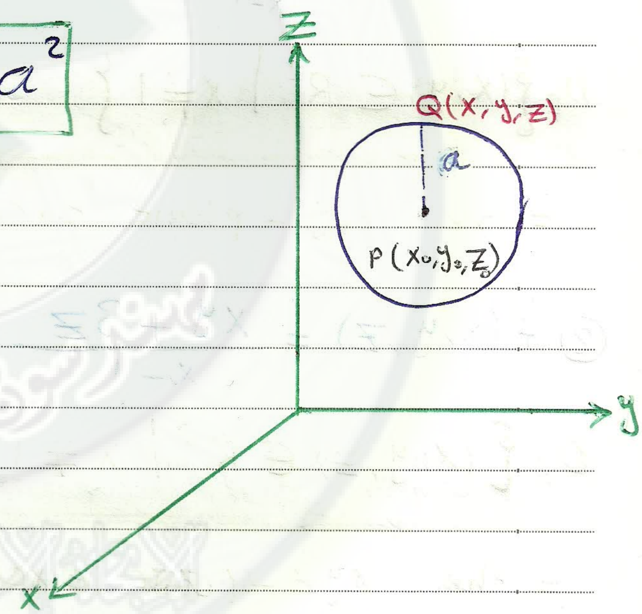
① $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

② The mid point of AB is:

$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} + \frac{z_1 + z_2}{2}\right)$

The equation of the sphere whose center (x_0, y_0, z_0) and radius a is:

$$\boxed{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2}$$

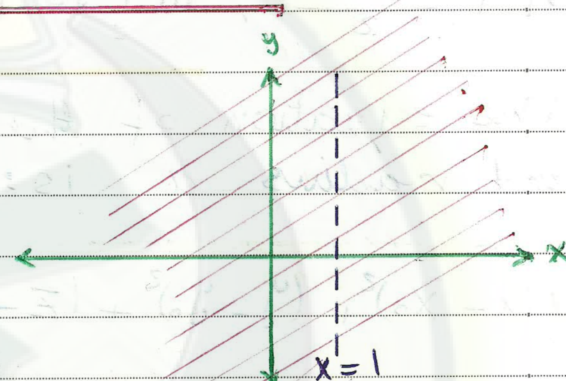


لا تنسونا من صالح دعائكم

* Partial Derivatives *

$y = f(x) \rightarrow$ function in one variable, its domain is a subset of R.
$Z = f(x,y) \rightarrow$ function in two variable, its domain is a subset of R^2.
$w = f(x,y,z) \rightarrow$ function in three variable, its domain is a subset of R^3.

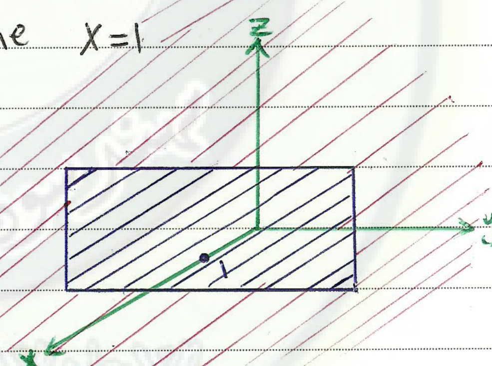
e.g: Find the domain of:



① $f(x,y) = \dfrac{xy}{x-1}$

$D_f \{(x,y) \in R^2 \mid x \neq 1\}$

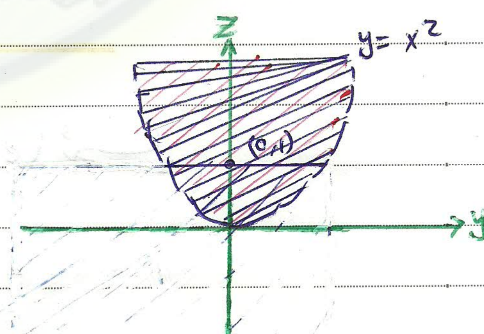
= the xy-plane except the line $x=1$



② $f(x,y,z) = \dfrac{xy+3z}{x-1}$

$D_f \{(x,y,z) \in R^3 \mid x \neq 1\}$

= the xyz-space except the points on the plane $x=1$



③ $f(x,y) = \ln(y - x^2)$

$y - x^2 > 0$

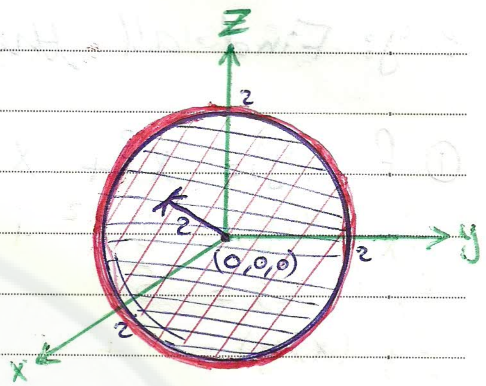
$y > x^2$

$y = x^2$

$D_f = \{(x,y) \in R^2 : y > x^2\}$

= the region of the points that are above the parabola $y = x^2$

④ $f(x,y,z) = \sqrt{4 - x^2 - y^2 - z^2}$



$4 - x^2 - y^2 - z^2 \geq 0$

$x^2 + y^2 + z^2 \leq 4$

$x^2 + y^2 + z^2 = 4$

sphere, center (0,0,0), radius = 2

$D_f = \{(x,y,z) \in R^3 \mid x^2 + y^2 + z^2 \leq 4\}$

= the boundary and the interior of the sphere $x^2 + y^2 + z^2 = 4$

Def: If $z = f(x,y)$ then the partial derivative of f w.r.t

① x is $f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$

② y is $f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$

e.g: By using the definition find the partial derivatives of: $f(x,y) = x^2 + 3xy - y + 4$

$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h)y - y + 4 - x^2 - 3xy + y - 4}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 + \cancel{3xy} + 3hy \cancel{- y} + \cancel{4} - \cancel{x^2} - \cancel{3xy} + \cancel{y} - \cancel{4}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 3y)}{\cancel{h}} = 2x + 3y$

$f_y(x,y) = 3x - 1$

e.g: Find all the partial derivatives of:

① $f(x,y) = x^3 + xy + 4y + 2$ at $(1,2)$

$f_x(x,y) = 3x^2 + y$

$f_x(1,2) = 5$

$f_y(x,y) = x + 4$

$f_y(1,2) = 5$

② $f(x,y,z) = x\sin(x^2y^3) + \cosh(xz) + \ln(y+z)$

$f_x(x,y,z) = 2x^2y^3\cos(x^2y^3) + \sin(x^2y^3) + z\sinh(xz)$

$f_y(x,y,z) = 3x^3y^2\cos(x^2y^3) + \dfrac{1}{y+z}$

$f_z(x,y,z) = x\sinh(xz) + \dfrac{1}{y+z}$

③ $f(x,y) = x^y + \tan^{-1}(2x+3y)$

$f_x(x,y) = y\,x^{y-1} + \dfrac{2}{1+(2x+3y)^2}$

$f_y(x,y) = x^y\ln x + \dfrac{3}{1+(2x+3y)^2}$

④ $f(\rho,\theta,\Phi) = \sqrt{\rho^2+3\Phi} + e^{\rho\theta}$

$f_\rho(\rho,\theta,\Phi) = \dfrac{2\rho}{2\sqrt{\rho^2+3\Phi}} + \theta e^{\rho\theta}$

$\dfrac{\partial f}{\partial\theta}(\rho,\theta,\Phi) = \rho e^{\rho\theta}$

$\dfrac{\partial f}{\partial\Phi}(\rho,\theta,\Phi) = \dfrac{3}{2\sqrt{\rho^2+3\Phi}}$

High_Order Partial derivatives.
(Mixed derivatives)

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} \qquad x \text{ then } y$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} \qquad y \text{ then } x$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad x \text{ then } x$$

$$\frac{\partial^3 f}{\partial z \partial y \partial x} = f_{xyz} \qquad x \text{ then } y \text{ then } z$$

$$f_{xy} = f_{yx}$$

$$f_{xxy} = f_{xyx} = f_{yxx}$$

e.g. If $f(x,y) = x^3 y^{-2} + x e^{2y}$, Find:

① $\frac{\partial^2 f}{\partial y \partial x}$: $\frac{\partial f}{\partial x} = 3x^2 y^{-2} + e^{2y}$

$$\frac{\partial^2 f}{\partial y \partial x} = -6x^2 y^{-3} + 2e^{2y}$$

② $\frac{\partial^2 f}{\partial x \partial y}$: $\frac{\partial f}{\partial y} = -2\, x^3 y^{-3} + 2\, x e^{2y}$

$$\frac{\partial^2 f}{\partial x \partial y} = -6\, x^2 y^{-3} + 2\, e^{2y} = \frac{\partial^2 f}{\partial y \partial x}$$

③ $\frac{\partial^3 f}{\partial y^3}$: $\frac{\partial^2 f}{\partial y^2} = 6\, x^3 y^{-4} + 4\, x e^{2y}$

$$\frac{\partial^3 f}{\partial y^3} = -24\, x^3 y^{-5} + 8\, x e^{2y}$$

e.g: If $f(x,y) = \tan^{-1}(x^2 y^3)$, Find:

f_{xy}: $f_x = \dfrac{2 x y^3}{1 + x^4 y^6}$

$$f_{xy} = \frac{(1 + x^4 y^6)(6 x y^2) - (2 x y^3)(6 x^4 y^5)}{(1 + x^4 y^6)^2}$$

Implicit Differentiation : الاشتقاق الضمني

e.g: Given that:

$$x^2 y + z^2 + x^3 z = x + 2y + 5$$

Find: ① $\dfrac{\partial z}{\partial x}$ نعتبر x و z هي المتغيرات والباقي ثوابت ، ② $\dfrac{\partial z}{\partial y}$ نعتبر z و y هي المتغيرات والباقي ثوابت

Solution:

① $2xy + 2z\dfrac{\partial z}{\partial x} + 3x^2 z + x^3 \dfrac{\partial z}{\partial x} = 1$

$$2z\frac{\partial z}{\partial x} + x^3\frac{\partial z}{\partial x} = 1 - 2xy - 3x^2 z$$

$$\frac{\partial z}{\partial x}(2z + x^3) = 1 - (2xy + 3x^2 z)$$

$$\frac{\partial z}{\partial x} = \frac{1 - (2xy + 3x^2 z)}{(2z + x^3)}$$

② $x^2 + 2z\dfrac{\partial z}{\partial y} + x^3\dfrac{\partial z}{\partial y} = 2$

$$2z\frac{\partial z}{\partial y} + x^3\frac{\partial z}{\partial y} = 2 - x^2$$

$$\frac{\partial z}{\partial y}(2z + x^3) = 2 - x^2$$

$$\frac{\partial z}{\partial y} = \frac{(2 - x^2)}{(2z + x^3)}$$

* $f(x,y,z)=0$

$$f_x * 1 + f_y * 0 + f_z * \frac{\partial z}{\partial x} = 0$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-f_x}{f_z}} \qquad \boxed{\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}}$$

* $f(x,y,z,w)=0$

$$\boxed{\frac{\partial w}{\partial x} = \frac{-f_x}{f_w}}$$

$$\boxed{\frac{\partial w}{\partial y} = \frac{-f_y}{f_w}}$$

$$\boxed{\frac{\partial w}{\partial z} = \frac{-f_z}{f_w}}$$

* $f(x,y)=0$

$$\boxed{\frac{\partial y}{\partial x} = \frac{-f_x}{f_y}}$$

e.g: Given that:

$$x^3 e^{2y} = \sin(xz^2) + \tan(yz)$$

Find: ① $\frac{\partial z}{\partial x}$, ② $\frac{\partial z}{\partial y}$

Solution:

① $\sin(xz^2) + \tan(yz) - x^3 e^{2y} = 0$

$$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} = \frac{-(z^2\cos(xz^2) - 3x^2 e^{2y})}{2xz\cos(xz^2) + y\sec^2(yz)}$$

② $$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z} = \frac{-(z\sec^2(yz) - 2x^3 e^{2y})}{2xz\cos(xz^2) + y\sec^2(yz)}$$

Ex: Given that:

$$x^3 \sin(x^2 z) = e^{xyz} \longrightarrow x^3\sin(x^2z) - e^{xyz} = 0$$

Find: ① $\frac{\partial z}{\partial x}$, ② $\frac{\partial z}{\partial y}$

Solution:

① $$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} = \frac{-(3x^2\sin(x^2z) + 2x^4 z\cos(x^2z) - yz e^{xyz})}{x^5\cos(x^2z) - xy e^{xyz}}$$

② $$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z} = \frac{-(-xz e^{xyz})}{x^5\cos(x^2z) - xy e^{xyz}}$$

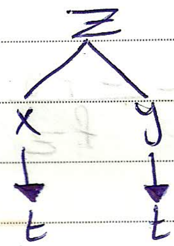
e.g: Given that:

$$\sinh(x^2y^3) = e^{xy} + y \quad , \text{ Find } \frac{\partial y}{\partial x}:$$

$$f(x,y) = \sinh(x^2y^3) - e^{xy} - y = 0$$

$$\frac{\partial y}{\partial x} = \frac{-f_x}{f_y} = \frac{-(2xy^3\cosh(x^2y^3) - ye^{xy})}{3x^2y^2\cosh(x^2y^3) - xe^{xy} - 1}$$

Chain Rule: قاعدة السلسلة



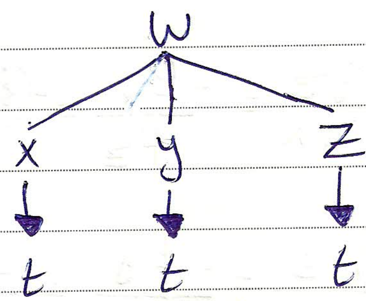
* $Z = f(x,y)$

$x = f(t)$

$y = g(t)$

Then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$



* $W = f(x,y,z)$

$x = x(t)$

$y = y(t)$

$z = z(t)$

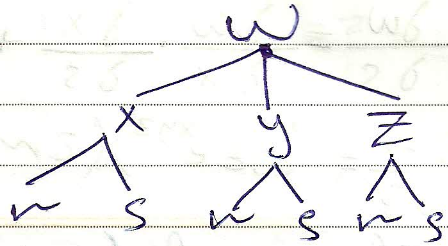
Then:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Chapter 1

Day 18 Month 2 Year (2009)

* $w = f(x, y, z)$
  $x = x(r, s)$
  $y = y(r, s)$
  $z = z(r, s)$



Then:

① $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

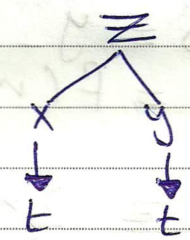
② $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

e.g: If $z = x^3 y^2$

$x = e^{2t}$

$y = \sin(t^2)$

Find $\frac{dz}{dt}$:



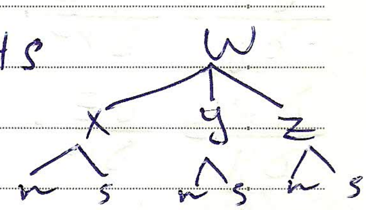
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = (3x^2y^2)(2e^{2t}) + (2x^3y)(2t\cos(t^2))$$

e.g: If $w = e^{xyz}$

$x = e^{rs^2}$, $y = \ln(r+s)$, $z = r + 4s$

Find: ① $\frac{\partial w}{\partial r}$ ② $\frac{\partial w}{\partial s}$



solution:

① $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial r} = (yz\,e^{xyz})(s^2 e^{rs^2}) + (xz\,e^{xyz})\left(\frac{1}{r+s}\right) + (xy\,e^{xyz})(1)$$

② Ex

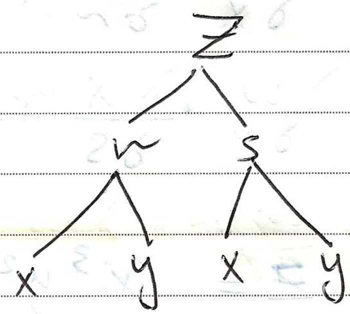
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial s} = \left(yz e^{xyz}\right)\left(2rs\, e^{rs^2}\right) + \left(xz e^{xyz}\right)\left(\frac{1}{r+s}\right) + \left(xy e^{xyz}\right)(4)$$

e.g: If $z = f(x-y,\ y-x)$

Show that:

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$



let: $r = x - y$

$s = y - x$

$z = f(r, s)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot(1) + \frac{\partial z}{\partial s}\cdot(-1)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} - \frac{\partial z}{\partial s} \longrightarrow ①$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial y}$$

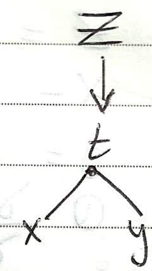
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot(-1) + \frac{\partial z}{\partial s}\cdot(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial s} - \frac{\partial z}{\partial r} \longrightarrow ②$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = ① + ② = \cancel{\frac{\partial z}{\partial r}} - \cancel{\frac{\partial z}{\partial s}} + \cancel{\frac{\partial z}{\partial s}} - \cancel{\frac{\partial z}{\partial r}} = 0$$

#

e.g: Given that:



$$Z = f^3(x^2 + y^2)$$

$$f(25) = 4$$

$$f'(25) = -1$$

Find: ① $\left.\dfrac{\partial Z}{\partial x}\right|_{(3,4)}$ ② $\left.\dfrac{\partial Z}{\partial y}\right|_{(3,4)}$

let: $t = x^2 + y^2$

$$Z = f^3(t)$$

① $$\frac{\partial Z}{\partial x} = \frac{dZ}{dt} \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial z}{\partial x} = 3f^2(t)\, f'(t) \cdot (2x)$$

$$\frac{\partial z}{\partial x} = 3f^2(x^2 + y^2)\, f'(x^2 + y^2)\,(2x)$$

$$\left.\frac{\partial z}{\partial x}\right|_{(3,4)} = 3f^2(9+16)\, f'(9+16)\,(6) = (3)(16)(-1)(6) = -308$$

② EX $$\frac{\partial Z}{\partial y} = \frac{dZ}{dt} \cdot \frac{\partial t}{\partial y}$$

$$\frac{\partial z}{\partial y} = 3f^2(t)\, f'(t) \cdot (2y)$$

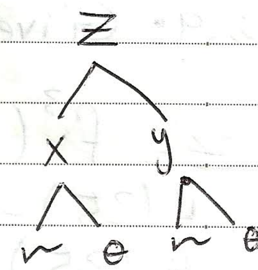
$$\frac{\partial Z}{\partial y} = 3f^2(x^2 + y^2)\, f'(x^2 + y^2) \cdot (2y)$$

$$\left.\frac{\partial Z}{\partial y}\right|_{(3,4)} = 3f^2(9+16)\, f'(9+16) \cdot (8) = (3)(16)(-1)(8) = -616$$

e.g: $z = f(x, y)$

$x = r\cos\theta$

$y = r\sin\theta$



Find: ① $\frac{\partial^2 f}{\partial r^2}$, ② $\frac{\partial^2 f}{\partial\theta\partial r}$, ③ $\frac{\partial^2 f}{\partial\theta^2}$

Solution:

① $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$

$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\cdot\sin\theta$

$\frac{\partial^2 z}{\partial r^2} = \left[\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\partial x}\sin\theta\right]\cos\theta + \left[\frac{\partial^2 z}{\partial x\partial y}\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin\theta\right]\sin\theta$

$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial y\partial x}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta$

$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{2\,\partial^2 z}{\partial y\partial x}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta$

---

② $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$

$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\cdot\sin\theta$

$\frac{\partial^2 z}{\partial\theta\partial r} =$

Chapter 1

Day 18 Month 2 Year (2009)

Chapter 1 Day 23 Month 2 Year (2009)

∇^2 : Laplace operator

$$\nabla^2 u_{(x,y)} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\nabla^2 u_{(x,y,z)} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Gradient الميل

Def:

① $\nabla f(x,y) = f_x \hat{i} + f_y \hat{j}$

② $\nabla f(x,y,z) = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

e.g: Find the gradient of the given functions:

① $f(x,y) = e^{2x} + \sin^{-1} y$ at $(\ln 2, 0)$
and then find $\| \nabla f(\ln 2, 0) \|$

$$\nabla f(x,y) = f_x \hat{i} + f_y \hat{j}$$
$$= (2e^{2x})\hat{i} + \frac{1}{\sqrt{1-y^2}} \hat{j}$$

$$\nabla f(\ln 2, 0) = 8\hat{i} + \hat{j}$$

$$\| \nabla f(\ln 2, 0) \| = \sqrt{64+1} = \sqrt{65}$$

② $f(x, y, z) = xy + xz + yz$ at $(1, 2, 3)$

$\nabla f(x,y,z) = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

$\nabla f(x,y,z) = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$

$\nabla f(1,2,3) = 5\hat{i} + 4\hat{j} + 3\hat{k}$

---

e.g: Given that: $f(x,y) = \frac{1}{2}x^2 + \frac{1}{3}y^3$

Find the $\nabla(\|\nabla f(x,y)\|)$

$\nabla f(x,y) = f_x \hat{i} + f_y \hat{j}$

$\nabla f(x,y) = x\hat{i} + y^2 \hat{j}$

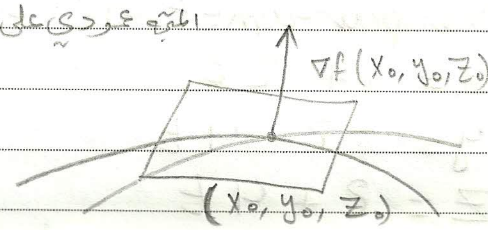
$\|\nabla f(x,y)\| = \sqrt{x^2 + y^4}$

$\nabla(\|\nabla f(x,y)\|) = \frac{x}{\sqrt{x^2 y^4}}\hat{i} + \frac{2y^3}{\sqrt{x^2+y^4}}\hat{j}$

$f(x, y, z) = 0$

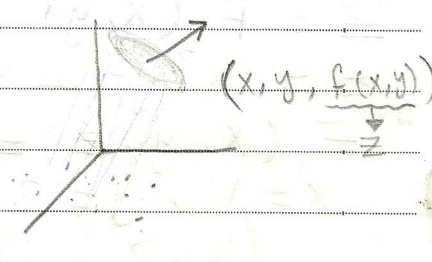
$\nabla f(x_0, y_0, z_0)$

المتجه عمودي على السطح



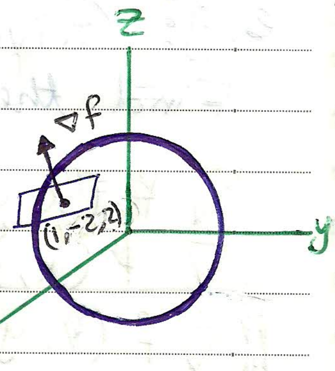
$z = f(x,y)$

$f(x,y,z) = f(x,y) - z = 0$



Tangent plane and normal line: معادلة مستوى مماس

e.g: Find the equation of the tangent plane and the parametric equations of the normal line to the surface of: $x^2+y^2+z^2=14$ at $(1,-2,2)$



$$f(x,y,z)=x^2+y^2+z^2-14=0$$

$$\nabla f(x,y,z)=2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}$$
$$\nabla f(1,-2,2)=2\,\hat{i}-4\,\hat{j}+4\,\hat{k}$$

The equation of the tangent plane is:

$$2(x-1)-4(y+2)+4(z-2)=0$$
$$x-1-2y-4+2z-4=0$$
$$x-2y+2z-9=0$$

The parametric equations of the normal line is:

$x=1+2t$

$y=-2-4t \qquad t\in R$

$z=2+4t$

---

e.g: Find the equation of the tangent plane to the surface of:

$z=f(x,y)$ where $f(x,y)=x^2y^3+x$ when $(x,y)=(1,2)$

$z=x^2y^3+x$

gor $F(x,y,z)=x^2y^3+x-z=0$

$x=1,\ y=2 \longrightarrow z=f(1,2)=9$

$(1,2,9)$

$$\nabla F(x, y, z) = (2xy^3 + 1)\hat{i} + (3x^2y^2)\hat{j} - \hat{k}$$

$$\nabla F(1,2,9) = 17\hat{i} + 12\hat{j} - \hat{k}$$

$\therefore$ The equation of the tangent plane is:

$$17(x-1) + 12(y-2) - (z-9) = 0$$

$$\left.\begin{aligned} x &= 1 + 17t \\ y &= 2 + 12t \qquad t \in R \\ z &= 9 - t \end{aligned}\right\}$$

The parametric equation of the normal line

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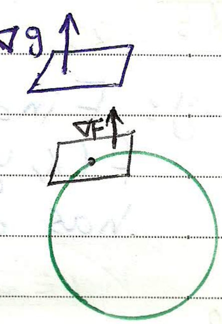
e.g: Find all the points on the surface of $x^2 + y^2 + z^2 = 9$ at which the tangent plane is parallel to the plane: $2x + 2y - z = 20$



$$f(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$$

$$g(x, y, z) = 2x + 2y - z - 20 = 0$$

$$\nabla f \parallel \nabla g$$

α: scalar

$$\nabla f = \alpha \nabla g$$

$$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \alpha(2\hat{i} + 2\hat{j} - \hat{k})$$

$$2x = 2\alpha \longrightarrow x = \alpha$$

$$2y = 2\alpha \longrightarrow y = \alpha$$

$$2z = -\alpha \longrightarrow z = \frac{-\alpha}{2}$$

$$\alpha^2 + \alpha^2 + \frac{\alpha^2}{4} = 9 \longrightarrow 4\alpha^2 + 4\alpha^2 + \alpha^2 = 36$$

$$9\alpha^2 = 36 \longrightarrow \alpha^2 = 4 \longrightarrow \alpha = \pm 2$$

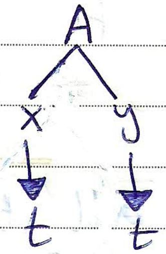
The required points are: $(2, 2, -1)$, $(-2, -2, 1)$

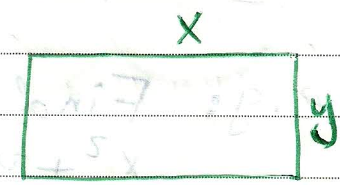
e.g: If a length of a rectangle is increasing at a rate of 2 cm/s while its width is decreasing at a rate of 1 cm/s.
Find the rate of chang of its area when its length = 8 cm, and its width = 6 cm

A = area

$\frac{dx}{dt} = 2 \text{ cm/s}$, $\frac{dy}{dt} = -1 \text{ cm/s}$, $\left.\frac{dA}{dt}\right|_{\substack{x=8 \\ y=6}} = ??$

$A = xy$





$$\frac{dA}{dt} = \frac{\partial A}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial A}{\partial y} \cdot \frac{dy}{dt}$$

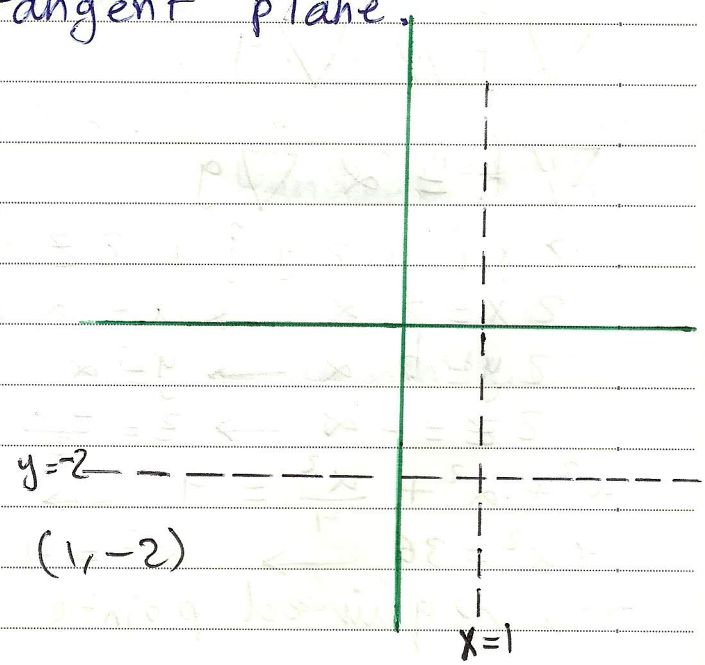
$$\frac{dA}{dt} = y \frac{dx}{dt} + x \frac{dy}{dt}$$

$$\left.\frac{dA}{dt}\right|_{(8,6)} = (6)(2) + (8)(-1) = 4 \text{ cm}^2/\text{s}$$

e.g: Find the point at which
$f(x,y) = x^2 + y^2 - 2x + 4y + 1$
has a horizantal tangent plane.

$f_x = 2x - 2$
$f_y = 2y + 4$

$f_x = 0 \to x = 1$
$f_y = 0 \to y = -2$



$\therefore$ the required point is $(1,-2)$

Directional Derivative: المشتقة المتجهة

If f is a function in two / three variables $X_0 = (x_0, y_0)$ / (x_0, y_0, z_0)

Then the directional derivative of f in the direction of the unit vector $\hat{u}$ is:

$$\boxed{D_{\hat{u}} f(X_0) = \nabla f(X_0) \cdot \hat{u}}$$

$f_{\hat{u}}(X_0)$: قد تكتب في بعض الكتب

e.g: Find the directional derivative of $f(x,y) = xy + 2x - 3y$ at $(1,-1)$ in the direction of $3\hat{i} - 4\hat{j}$

$\vec{a} = 3\hat{i} - 4\hat{j}$

$\|\vec{a}\| = \sqrt{9+16} = 5$

$\hat{u} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$

$\nabla f(x,y) = (y+2)\hat{i} + (x-3)\hat{j}$

$\nabla f(1,-1) = \hat{i} - 2\hat{j}$

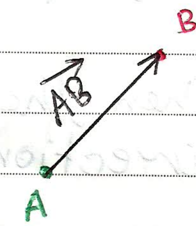
$\therefore D_{\hat{u}} f(1,-1) = \nabla f(1,-1) \cdot \hat{u} = (\hat{i} - 2\hat{j}) \cdot (\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j})$

$D_{\hat{u}} f(1,-1) = \frac{3}{5} + \frac{8}{5} = \frac{11}{5}$

e.g: Find the directional derivative of
$f(x,y,z) = xy + xz \ , \ yz$
at $A(1,-3,1)$ towards $B(3,-1,0)$

$\overrightarrow{AB} = 2\hat{i} + 2\hat{j} - \hat{k}$

$\|\overrightarrow{AB}\| = \sqrt{4+4+1} = 3$



$\therefore \hat{u} = \dfrac{1}{\|\overrightarrow{AB}\|}\overrightarrow{AB} = \dfrac{2}{3}\hat{i} + \dfrac{2}{3}\hat{j} - \dfrac{1}{3}\hat{k}$

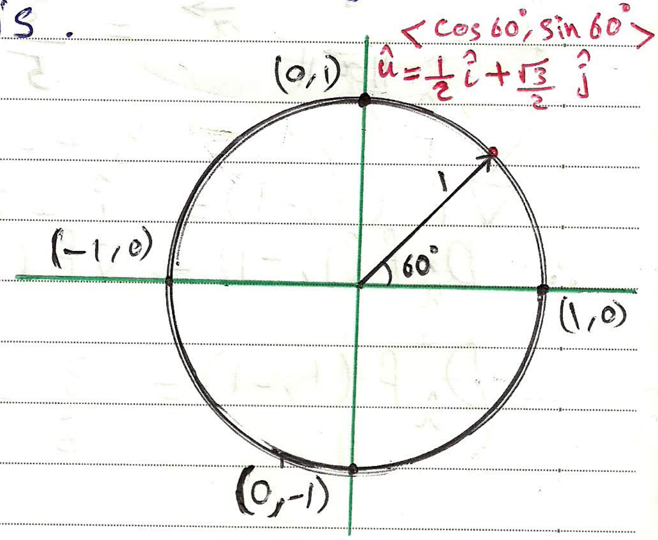
$\nabla f(x,y,z) = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$

$\nabla f(1,-3,1) = -2\hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore D_{\hat{u}} f(1,-3,1) = \nabla f(1,-3,1) \cdot \hat{u}$

$D_{\hat{u}} f(1,-3,1) = \dfrac{-4}{3} + \dfrac{4}{3} + \dfrac{2}{3} = \dfrac{2}{3}$

e.g: find the directional derivative of
$f(x,y) = x^2 + 3y + 5x$ at $(1,2)$
in the direction that makes an angle of 60°
with the positive x- axis.



$\hat{u} = \dfrac{1}{2}\hat{i} + \dfrac{\sqrt{3}}{2}\hat{j}$

$\nabla f(x,y) = (2x+5)\hat{i} + 3\hat{j}$

$\nabla f(1,2) = 7\hat{i} + 3\hat{j}$

$D_{\hat{u}} f(1,2) = \nabla f(1,2) \cdot \hat{u}$

$D_{\hat{u}} f(1,2) = \dfrac{7 + 3\sqrt{3}}{2}$

e.g: Given that:

$D_{\hat{u}} f(x_0, y_0) = 1$

$\nabla f(x_0, y_0) = 2\hat{i} - \hat{j}$

Find all the possible values of the unit vector $\hat{u}$.

$D_{\hat{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}$ let $\hat{u} = u_1 \hat{i} + u_2 \hat{j}$

$1 = (2i - \hat{j}) \cdot (u_1 \hat{i} + u_2 \hat{j})$

$1 = 2u_1 - u_2 \longrightarrow ①$

But: $\|\hat{u}\|^2 = 1^2$ بتربيع الطرفين

$u_1^2 + u_2^2 = 1 \longrightarrow ②$

① in ② $u_1^2 + (2u_1 - 1)^2 = 1 \rightarrow u_1^2 + 4u_1^2 - 4u_1 + 1 = 1$

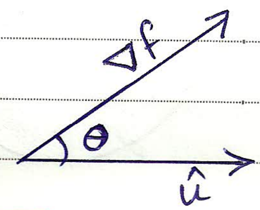
$5u_1^2 - 4u_1 = 0 \longrightarrow u_1(5u_1 - 4) = 0$

$(u_1 = 0 \longrightarrow u_2 = -1)$: $\hat{u} = -\hat{j}$ or $(u_1 = \frac{4}{5} \rightarrow u_2 = \frac{3}{5})$: $\hat{u} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$

$D_{\hat{u}} f = \nabla f \cdot \hat{u}$

$= \|\nabla f\| \, \|\hat{u}\| \cos\theta$

$= \|\nabla f\| \cos\theta$



Notes:

① If $\nabla f \neq 0$ at fixed point. Then the maximum value of all directional derivatives is $\|\nabla f\|$ and it occurs in the direction ∇f.

② and among all the directional derivatives, smallest value is $-\|\nabla f\|$ and it occurs in the opposite direction of ∇f $(-\nabla f)$.

③ $D_{\hat{i}} f = \nabla f \cdot \hat{i} = (f_x \hat{i} + f_y \hat{j} + f_z \hat{k}) \cdot \hat{i} = f_x$

$D_{\hat{j}} f = f_y$

$D_{\hat{k}} f = f_z$

$D_{-\hat{i}} f = -f_x$

Max._ Min. Problems of a function in two variable

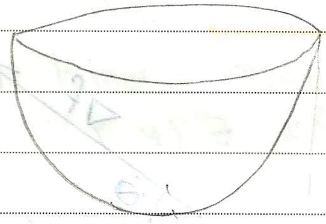
Def: A point (x_0, y_0) is called a critical point of $f(x,y)$ if:

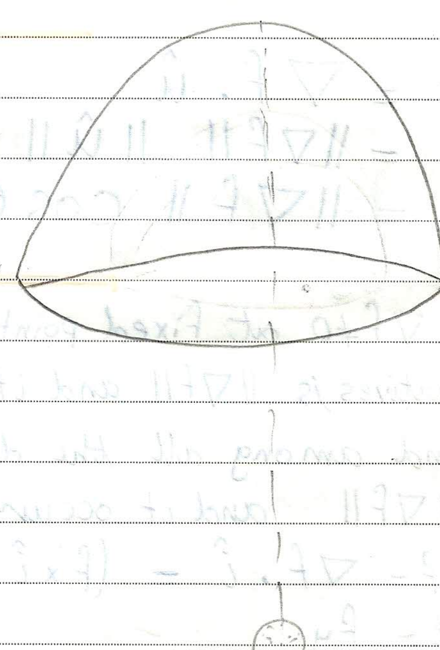
$$\boxed{f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0}$$

or:

$$\boxed{f_x(x_0, y_0) \text{ d.n.e} \quad \text{or} \quad f_y(x_0, y_0) \text{ d.n.e}}$$

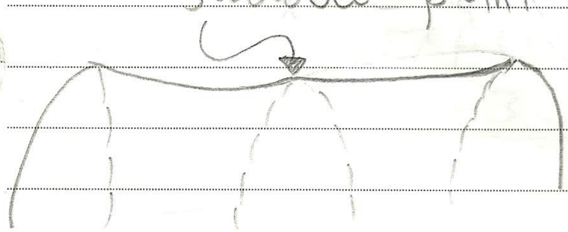
relative max





relative min.

saddle point



2^{nd} Partials Test:

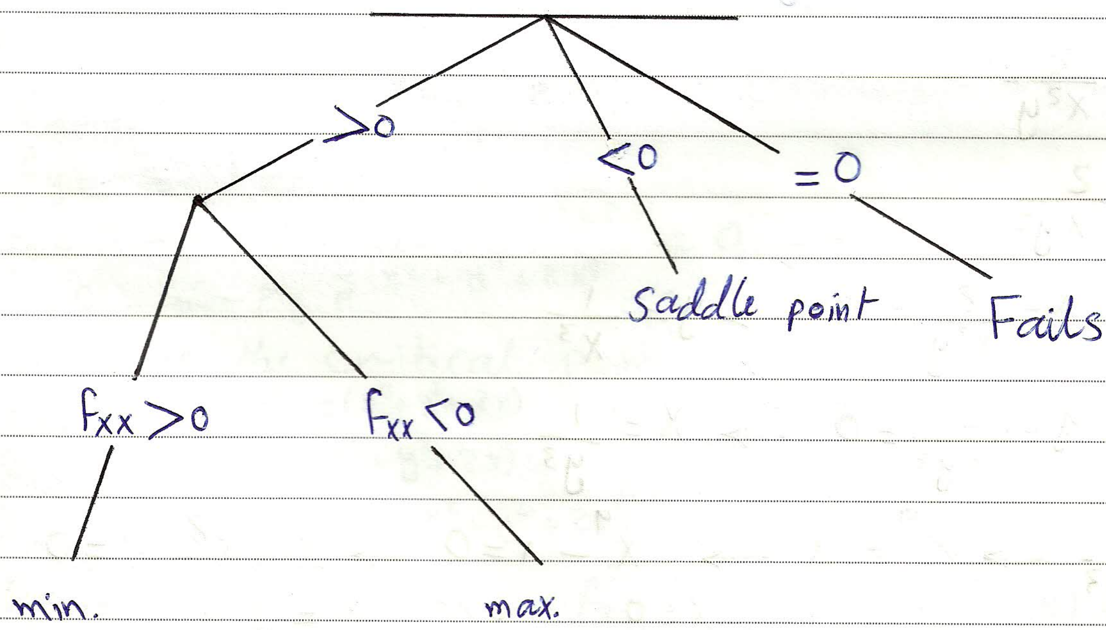
If (x_0, y_0) is a critical point of $f(x,y)$

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

Then:

1. $f(x_0, y_0)$ is a relative max. if $D > 0$ and $f_{xx}(x_0, y_0) < 0$
2. $f(x_0, y_0)$ is a relative min. if $D > 0$ and $f_{xx}(x_0, y_0) > 0$
3. f has a saddle point at (x_0, y_0) if $D < 0$
4. inconclusive if $D = 0$

$$D = f_{xx} f_{yy} - f_{xy}^2$$



e.g: Locate all relative maxima, relative minima and saddle point (if any):

① $f(x,y) = y^2 + xy + 3y + 2x + 3$

$f_x = y + 2$

$f_y = 2y + x + 3$

$f_x = 0 \longrightarrow y = -2$

$f_y = 0 \longrightarrow 2y + x + 3 = 0 \longrightarrow -4 + x + 3 = 0 \longrightarrow x = 1$

$\therefore (1,-2)$ is the critical point.

$f_{xx} = 0$

$f_{yy} = 2$

$f_{xy} = 1$

$D = f_{xx}(1,-2)\, f_{yy}(1,-2) - f_{xy}^2(1,-2) = (0)(2) - 1^2 = -1 < 0$

$\therefore f$ has a saddle point at $(1,-2)$.

② $f(x,y) = x^2 + y^2 + \dfrac{2}{xy}$

$f_x = 2x - \dfrac{2}{x^2 y}$

$f_y = 2y - \dfrac{2}{x y^2}$

$f_x = 0 \longrightarrow 2x - \dfrac{2}{x^2 y} = 0 \longrightarrow y = \dfrac{1}{x^3}$

$f_y = 0 \longrightarrow 2y - \dfrac{2}{x y^2} = 0 \longrightarrow x = \dfrac{1}{y^3}$

$x = \dfrac{1}{\left(\frac{1}{x^3}\right)^3} \longrightarrow x^9 = x \longrightarrow x^9 - x = 0 \longrightarrow x(x^8 - 1) = 0$

$x = 0$ or $x = \pm 1$

$x = 0$ rejected لأنها خارج المجال.

$x = 1 \rightarrow y = 1 \rightarrow (1,1)$ is a critical point

$x = -1 \rightarrow y = -1 \rightarrow (-1,-1)$ is a critical point

$f_{xx} = 2 + \frac{4}{x^3 y}$

$f_{yy} = 2 + \frac{4}{x y^3}$

$f_{xy} = \frac{2}{x^2 y^2}$

(x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx} f_{yy} - f_{xy}^2$	Conclusion
$(1, 1)$	$6 > 0$	6	2	$32 > 0$	min. $= f(1,1) = 4$
$(-1, -1)$	$6 > 0$	6	2	$32 > 0$	min. $= f(-1,-1) = 4$

③ $f(x, y) = e^x \sin y$

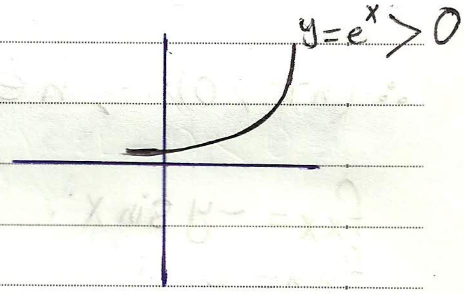
$f_x = e^x \sin y$

$f_y = e^x \cos y$

$f_x = 0 \longrightarrow e^x \sin y = 0 \longrightarrow \sin y = 0$

$f_y = 0 \longrightarrow e^x \cos y = 0 \longrightarrow \cos y = 0$

No critical points



$\cos^2 y + \sin^2 y = 1$

$0 + 0 = 1$!!??

④ $f(x, y) = e^{-(x^2 + y^2 + 2x)}$

$f_x = -(2x+2)\, e^{-(x^2 + y^2 + 2x)}$

$f_y = -(2y)\, e^{-(x^2 + y^2 + 2x)}$

$f_x = 0 \longrightarrow -(2x+2)\, e^{-(x^2+y^2+2x)} = 0 \longrightarrow 2x + 2 = 0 \longrightarrow x = -1$

$f_y = 0 \longrightarrow -2y\, e^{-(x^2+y^2+2x)} = 0 \longrightarrow 2y = 0 \longrightarrow y = 0$

$(-1, 0)$ is the critical point.

$f_{xx} = (2x+2)^2 e^{-(x^2+y^2+2x)} - 2 e^{-(x^2+y^2+2x)}$

$f_{yy} = 4y^2 e^{-(x^2+y^2+2x)} - 2 e^{-(x^2+y^2+2x)}$

$f_{xy} = 2y(2x+2)\, e^{-(x^2+y^2+2x)}$

$D = f_{xx}(-1,0)\, f_{yy}(-1,0) - f_{xy}^2(-1,0) = (-2e)(-2e) - 0 = 4e^2 > 0$

$f_{xx}(-1,0) = -2e < 0 \quad \therefore f(-1,0) = e$ is a relative max.

⑤ $f(x,y) = y \sin x$

$f_x = y \cos x$
$f_y = \sin x$

$f_x = 0 \rightarrow y \cos x = 0$
$f_y = 0 \rightarrow \sin x = 0 \rightarrow x = n\pi \quad , n \in \mathbb{Z}$
$\rightarrow y \cos n\pi = 0 \qquad \cos n\pi = (-1)^n$
$\rightarrow y = 0$

$\therefore (n\pi, 0) \quad , n \in \mathbb{Z}$ are the critical point.

$f_{xx} = -y \sin x$
$f_{yy} = 0$
$f_{xy} = \cos x$

$D = f_{xx}(n\pi, 0)\, f_{yy}(n\pi, 0) - f_{xy}^2(n\pi, 0)$
$(0)(0) - [\cos n\pi]^2 = -1 < 0$

$\therefore f$ has saddle point at $(n\pi, 0) \quad , n \in \mathbb{Z}$

Lagrange Multiplies مضروبات لاغرانج

To maximize or minimize $f(xy)$ (or $f(x,y,z)$) subject to the constraint (تخضع لقيود (تشرط))
$g(x,y)=0$ (or $g(x,y,z)=0$)

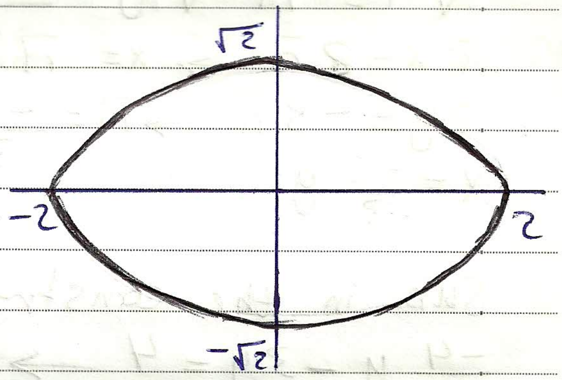
put $\nabla f = \lambda \nabla g$ $(\nabla f \parallel \nabla g)$

e.g: maximize and minimize:
$f(x,y) = xy$ subject to the constraint $2x^2+4y^2=8$
$g(x,y) = 2x^2+4y^2-8=0$

$\nabla f = y\hat{i} + x\hat{j}$

$\nabla g = 4x\hat{i} + 8y\hat{j}$

$\nabla f = \lambda \nabla g \rightarrow y\hat{i}+x\hat{j} = 4x\lambda\hat{i}+8y\lambda\hat{j}$



$y = 4x\lambda \rightarrow \lambda = \frac{y}{4x}$

$x = 8y\lambda \rightarrow \lambda = \frac{x}{8y}$

$\frac{y}{4x} = \frac{x}{8y} \rightarrow 8y^2 = 4x^2 \rightarrow x^2 = 2y^2$

فقط للتوضيح

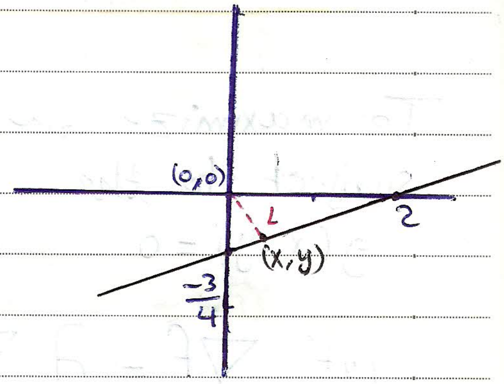
Sub. in the constraint:
$2(2y^2)+4y^2-8=0 \rightarrow 8y^2=8 \rightarrow \boxed{y=\pm 1} \rightarrow x^2=2 \rightarrow \boxed{x=\pm\sqrt{2}}$

$(\sqrt{2},1), (\sqrt{2},-1), (-\sqrt{2},1), (-\sqrt{2},-1)$

$f(\sqrt{2},1) = f(-\sqrt{2},-1) = \sqrt{2}$ max.

$f(-\sqrt{2},1) = f(\sqrt{2},-1) = -\sqrt{2}$ min.

e.g: Find the closest point on the line $2x-3y=4$ to the origin.



$$L=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}$$

$$f(x,y)=L^2=x^2+y^2$$

$$g(x,y)=2x-3y-4=0$$

$$\nabla f=2x\hat{i}+2y\hat{j}$$

$$\nabla g=2\hat{i}-3\hat{j}$$

$$\nabla f=\lambda\nabla g \longrightarrow 2x\hat{i}+2y\hat{j}=2\lambda\hat{i}-3\lambda\hat{j}$$

$$2x=2\lambda \longrightarrow x=\lambda$$

$$2y=-3\lambda \longrightarrow \lambda=\frac{-2}{3}y$$

$$x=\frac{-2}{3}y$$

Sub. in the constraint:

$$\frac{-4}{3}y-3y=4 \longrightarrow -4y-9y=12 \longrightarrow \boxed{y=\frac{-12}{3}} \longrightarrow \boxed{x=\frac{8}{13}}$$

$\therefore$ The closest point on the line is $\left(\frac{8}{13},\frac{-12}{13}\right)$

e.g: Maximize and minimize $f(x,y,z)=3x+6y+2z$ Subject to the constraint $2x^2+4y^2+z^2=70$.

$g(x,y,z)=2x^2+4y^2+z^2-70=0$

$\nabla f=3\hat{i}+6\hat{j}+2\hat{k}$

$\nabla g=4x\hat{i}+8y\hat{j}+2z\hat{k}$

$\nabla f=\lambda\nabla g$

$3=4\lambda x \longrightarrow \lambda=\frac{3}{4x}$

$6=8\lambda y \longrightarrow \lambda=\frac{3}{4y}$

$2=2\lambda z \longrightarrow \lambda=\frac{1}{z}$

$\frac{3}{4x}=\frac{3}{4y}=\frac{1}{z} \longrightarrow \frac{4x}{3}=\frac{4y}{3}=z \longrightarrow y=x \longrightarrow z=\frac{4x}{3}$

sub. in the constraint:

$2x^2+4x^2+\frac{16x^2}{9}=70 \longrightarrow 18x^2+36x^2+16x^2=70*9$

$70x^2=70*9 \longrightarrow x^2=9 \longrightarrow x=\pm 3$

$x=3 \longrightarrow y=3 \longrightarrow z=4$

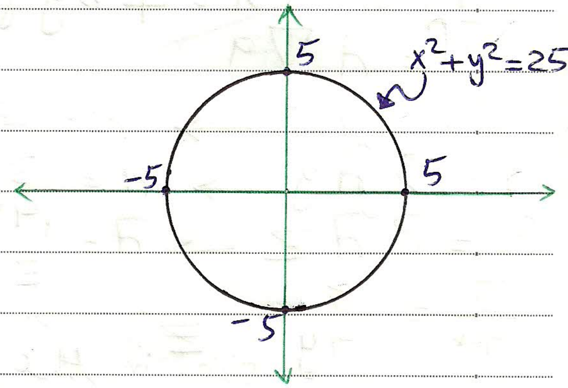
$x=-3 \longrightarrow y=-3 \longrightarrow z=-4$

$f(3,3,4)=35=\text{max}.$

$f(-3,-3,-4)=-35=\text{min}.$

Ex: Suppose that the temperature at a point (x,y) in a metal plate is $T(x,y) = 4x^2 + 4xy + y^2$. An ant, walking on the plate, traverses a circle centered at the origin with radius (5). Find the highest and lowest temperature encountered by the ant.

$g(x,y) = x^2 + y^2 - 25 = 0$



$\nabla T = (8x+4y)\hat{i} + (4x+2y)\hat{j}$

$\nabla g = 2x\hat{i} + 2y\hat{j}$

$\nabla T = \lambda \nabla g$

$8x + 4y = 2\lambda x \longrightarrow \lambda = \dfrac{4x+2y}{x}$

$4x + 2y = 2\lambda y \longrightarrow \lambda = \dfrac{2x+y}{y}$

$\dfrac{4x+2y}{x} = \dfrac{2x+y}{y} \longrightarrow x(2x+y) = 2y(2x+y)$

$(2x+y)(x-2y) = 0$ sub. in the constraint:

$2x + y = 0 \rightarrow y = -2x \longrightarrow x^2 + 4x^2 = 25 \rightarrow x^2 = 5 \rightarrow x = \pm 5$

or $x - 2y = 0 \rightarrow x = 2y \longrightarrow 4y^2 + y^2 = 25 \rightarrow y^2 = 5 \rightarrow y = \pm 5$

$x = 5 \longrightarrow y = -10$

$x = -5 \longrightarrow y = 10$

$y = 5 \longrightarrow x = 10$

$y = -5 \rightarrow x = -10$

$f(5,-10) = f(-5,10) = 0 = \min.$

$f(10,5) = f(-10,-5) = 625 = \max.$

Ex: Find the point on the plane $4x+3y+z=2$ that is closest to the point $(1,-1,1)$.

$L=\sqrt{(x-1)^2+(y+1)^2+(z-1)}$

$L=\sqrt{x^2-2x+1+y^2+2y+1+z^2-2z+1}$

$L^2=x^2-2x+y^2+2y+z^2-2z+3$

$f(x,y,z)=L^2=x^2-2x+y^2+2y+z^2-2z+3$

$g(x,y,z)=4x+3y+z-2=0$

$\nabla f=(2x-2)\hat{i}+(2y+2)\hat{j}+(2z-2)\hat{k}$

$\nabla g=4\hat{i}+3\hat{j}+\hat{k}$

$\nabla f=\lambda\nabla g$

$2x-2=4\lambda \rightarrow \lambda=\frac{x-1}{2}$

$2y+2=3\lambda \rightarrow \lambda=\frac{2y+2}{3}$

$2z-2=\lambda$

$\frac{x-1}{2}=\frac{2y+2}{3}=2z-2$

$\frac{x-1}{2}=2z-2 \rightarrow x-1=4z-4 \rightarrow x=4z-3$

$\frac{2y+2}{3}=2z-2 \rightarrow 2y+2=6z-6 \rightarrow 2y=6z-8 \rightarrow y=3z-4$

sub. in the constraint:

$4(4z-3)+3(3z-4)+z-2=0$

$16z-12+9z-12+z-2=0 \rightarrow 26z=26 \rightarrow z=1$

$\therefore$ The closest point is: $(1,-1,1)$

Ex: Maximize and/or minimize:

① $f(x,y) = x^2 - y$ subject to $x^2 + y^2 = 25$.

$g(x,y) = x^2 + y^2 - 25 = 0$

$\nabla f = 2x\hat{i} - \hat{j}$

$\nabla g = 2x\hat{i} - 2y\hat{j}$

$\nabla f = \lambda \nabla g$

$2x\hat{i} - \hat{j} = 2\lambda x\hat{i} - 2\lambda y\hat{j}$

$2x = 2\lambda x \rightarrow \lambda = 1$

$-1 = 2\lambda y \rightarrow y = \frac{-1}{2}$

Sub. in the Constraint:

$x^2 + \frac{1}{4} = 25 \rightarrow 4x^2 + 1 = 100 \rightarrow 4x^2 = 99 \rightarrow x^2 = \frac{99}{4}$

$x = \frac{+\sqrt{99}}{2} \longrightarrow y = \frac{-1}{2}$

$x = \frac{-\sqrt{99}}{2} \longrightarrow y = \frac{-1}{2}$

$f\left(\frac{\sqrt{99}}{2}, \frac{-1}{2}\right) = f\left(-\frac{\sqrt{99}}{2}, \frac{-1}{2}\right) = \frac{101}{4} = \max.$

② $f(x,y,z) = 2x + y - 2z$ subject to $x^2 + y^2 + z^2 = 4$.

$g(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$

$\nabla f = 2\hat{i} + \hat{j} - 2\hat{k}$, $\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla f = \lambda \nabla g \longrightarrow 2\hat{i} + \hat{j} - 2\hat{k} = 2\lambda x\hat{i} + 2\lambda y\hat{j} + 2\lambda z\hat{k}$

$2 = 2\lambda x \rightarrow \lambda = \frac{1}{x}$

$1 = 2\lambda y \rightarrow \lambda = \frac{1}{2y}$

$-2 = 2\lambda z \rightarrow \lambda = \frac{-1}{z}$

$\frac{1}{x} = \frac{1}{2y} = \frac{-1}{z}$

$x = -z$

$2y = -z \rightarrow y = \frac{-z}{2}$

sub. in the constraint:

$z^2 + \frac{z^2}{4} + z^2 = 4 \rightarrow 4z^2 + z^2 + 4z^2 = 16$

$9z^2 = 16 \rightarrow z = \frac{\pm 4}{3}$

$z = \frac{4}{3} \rightarrow y = \frac{-4}{6} \rightarrow x = \frac{-4}{3}$

$z = \frac{-4}{3} \rightarrow y = \frac{4}{6} \rightarrow x = \frac{4}{3}$

$f\left(\frac{-4}{3}, \frac{-4}{6}, \frac{4}{3}\right) = -6 = \min.$

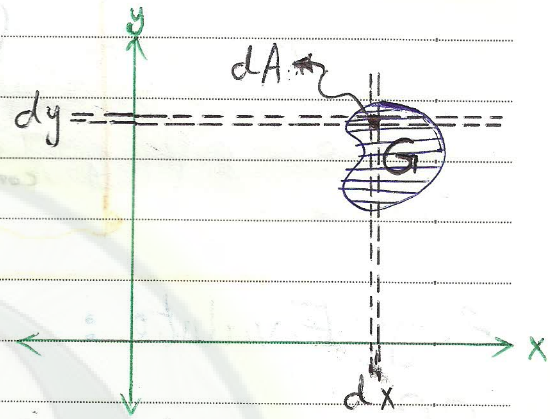
$f\left(\frac{4}{3}, \frac{4}{6}, \frac{-4}{3}\right) = 6 = \max.$

Chapter 2

Day 11 Month 3 Year (2009)

Multiple Integral

* Double Integral :



$$\iint_G f(x,y)\, dA$$

$$dA = dy\,dx = dx\,dy$$

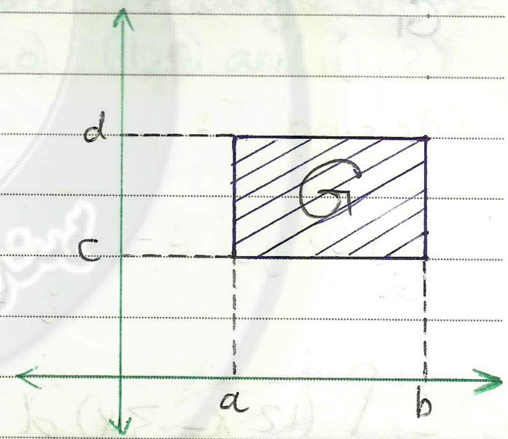
مستطيل

G: rectangular region:

$$\iint_G f(x,y)\, dA$$

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx$$

① ②



$$= \int_c^d \int_a^b f(x,y)\, dx\, dy$$

① ②

$$\boxed{\int_{\text{constant}}^{\text{constant}} \int_{\text{constant}}^{\text{constant}}}$$

Chapter 2

Day 11 Month 3 Year (2009)

Non-rectangular region:

$$\int_{\text{constant}}^{\text{constant}} \int_{?}^{?}$$

e.g: Evalute:

$$\iint_G 6xy\,dA \quad \text{where: } G = \{(x,y) : 0 \leq x \leq 1,\ -1 \leq y \leq 2\}$$

or:

$$G = [0,1] \times [-1,2]$$

Method I:

$$\iint_G 6xy\,dA = \int_0^1 \int_{-1}^2 6xy\,dy\,dx = \int_0^1 3xy^2 \Big]_{-1}^2 dx$$

$$= \int_0^1 (12x - 3x)\,dx = \int_0^1 9x\,dx = \frac{9x^2}{2}\Big]_0^1 = \boxed{\frac{9}{2}}$$

Method II:

$$\iint_G 6xy\,dA = \int_{-1}^2 \int_0^1 6xy\,dx\,dy = \int_{-1}^2 3x^2y \Big]_0^1 dy$$

$$= \int_{-1}^2 3y\,dy = \frac{3y^2}{2}\Big]_{-1}^2 = \frac{12}{2} - \frac{3}{2} = \boxed{\frac{9}{2}}$$

e.g: $\iint_R e^{x+y}\,dA$ where $R=[0,\ln 2]\times[0,\ln 3]$

$$\int_0^{\ln 2}\int_0^{\ln 3} e^{x+y}\,dy\,dx = \int_0^{\ln 2} e^x e^y\Big]_0^{\ln 3} dx = \int_0^{\ln 2} 3e^x - e^x\,dx = \int_0^{\ln 2} 2e^x\,dx$$

$$= 2e^x\Big]_0^{\ln 2} = 2\left[e^{\ln 2}-e^0\right] = 2(2-1) = 2$$

e.g: Evalute:

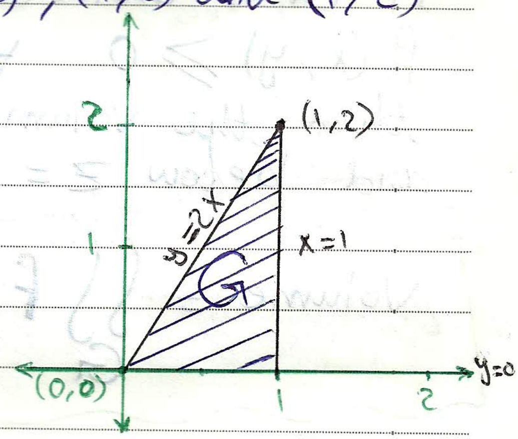
$\iint_G (2x+6y)\,dA$ when G is the triangle whose verticals (0,0), (1,0) and (1,2)

Method I

$$\iint_G (2x+6y)\,dA = \int_0^1\int_0^{2x}(2x+6y)\,dy\,dx$$

$$= \int_0^1 (2xy+3y^2)\Big]_0^{2x} dx = \int_0^1 (4x^2+12x^2-0)\,dx$$

$$= \int_0^1 16x^2\,dx = \frac{16x^3}{3}\Big]_0^1 = \frac{16}{3}$$

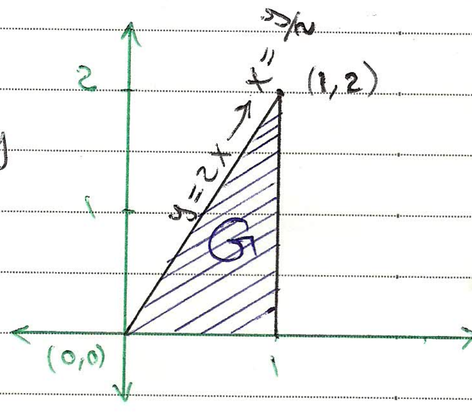


Method II

$$\iint_G (2x+6y)\,dA = \int_0^2\int_{\frac{y}{2}}^1 (2x+6y)\,dx\,dy$$

$$= \int_0^2 (x^2+6yx)\Big]_{\frac{y}{2}}^1 dy = \int_0^2 (1+6y)-\left(\frac{y^2}{4}+3y^2\right) dy$$

$$= \int_0^2 -\frac{13}{4}y^2+6y+1\,dy = -\frac{13}{12}y^3+3y^2+y\Big]_0^2 = \frac{16}{3}$$



Area using double Integration:

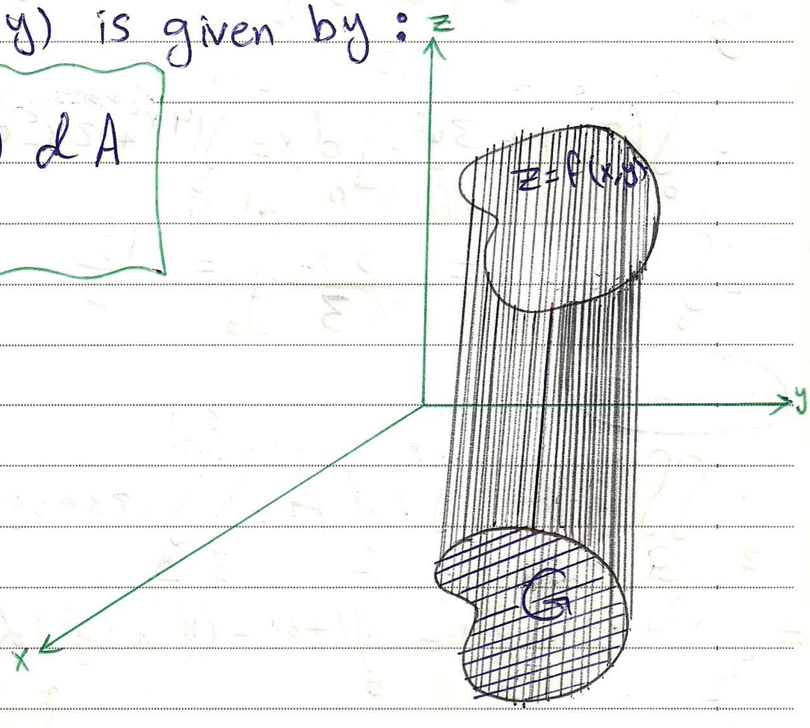
If G is a closed bounded region in the xy-plane (منطقة مغلقة ومحدودة) the area of G is given by:

$$\boxed{\text{Area} = \iint_G dA}$$

Volume using double Integration:

If G is a closed bounded region in the xy-plane
$f(x, y) \geq 0 \quad \forall (x, y) \in G$
then the volume of the solid (الجسم) that is above G and below $z = f(x, y)$ is given by:

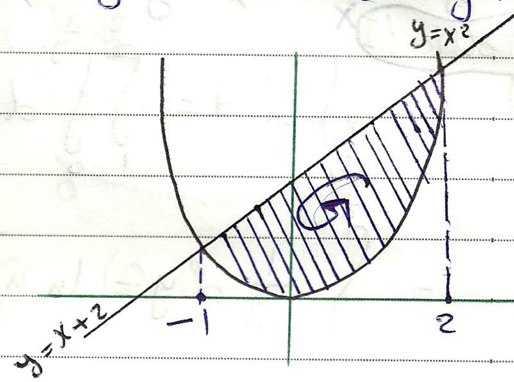
$$\boxed{\text{Volume} = \iint_G f(x, y)\, dA}$$



Chapter 2

Day 16 Month 3 Year (2009)

e.g: Find the area of the region enclosed by:



① $y = x^2$, $y = x+2$

Method I:

Area of $G = \iint\limits_G dA$

$$= \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2} (x+2)-(x^2)\,dy$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big]_{-1}^{2} = \left(6-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)$$

$$= \left(\frac{18-8}{3}\right) - \left(\frac{3-12+2}{6}\right) = \frac{20+7}{6} = 4.5$$

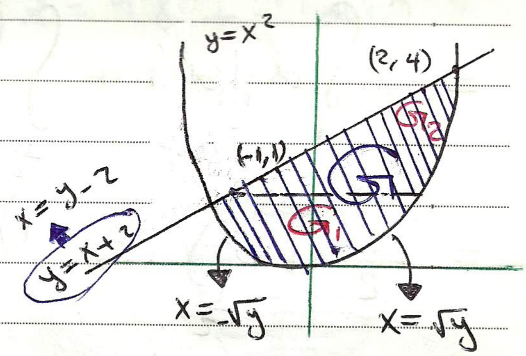
$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\boxed{x=2}$ or $\boxed{x=-1}$

Method II:



$$Area = \iint\limits_G dA = \iint\limits_{G_1} dA + \iint\limits_{G_2} dA$$

$$= \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4\int_{y-2}^{\sqrt{y}} dx\,dy$$

$$= \int_0^1 2\sqrt{y}\,dy + \int_1^4 \sqrt{y} - y + 2\,dy = 2*\frac{2}{3}y^{\frac{3}{2}}\Big]_0^1 + \frac{2}{3}y^{\frac{3}{2}} - \frac{y^2}{2} + 2y\Big]_1^4$$

$$= \frac{4}{3} + \frac{2}{3}*8 \cancel{-8} \cancel{+8} - \frac{2}{3} + \frac{1}{2} - 2 = 4.5$$

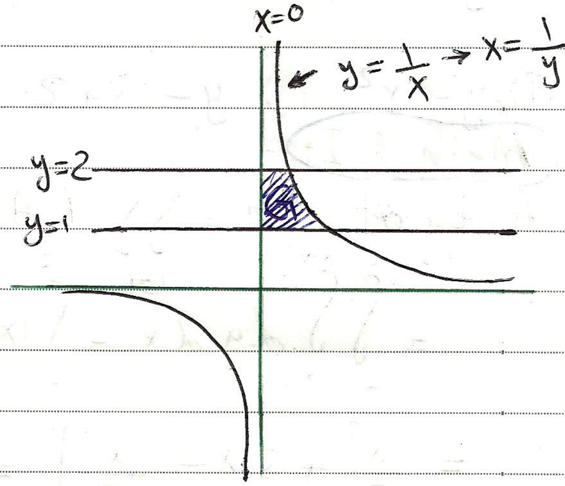
Hassooon ^_^

② $y=\frac{1}{x}$, $y=1$, $y=2$, $x=0$

Method I

$$\text{Area}=\iint_G dA=\int_1^2\int_0^{\frac{1}{y}} dx\,dy$$

$$=\int_1^2 \frac{1}{y}\,dy=\ln y\Big]_1^2=\ln 2$$

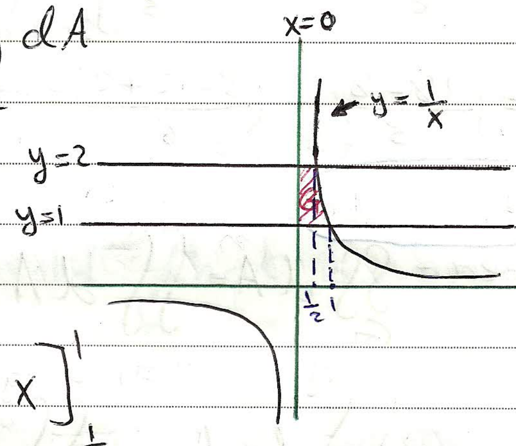


Method II

$$\text{Area}=\iint_G dA=\iint_{G_1} dA+\iint_{G_2} dA$$

$$=\int_0^{\frac{1}{2}}\int_1^2 dy\,dx+\int_{\frac{1}{2}}^1\int_1^{\frac{1}{x}} dy\,dx$$

$$=\frac{1}{2}+\int_{\frac{1}{2}}^1\left(\frac{1}{x}-1\right)dx=\ln x-x\Big]_{\frac{1}{2}}^1$$



$$=\frac{1}{2}+\ln 1-1-\ln\frac{1}{2}+\frac{1}{2}=-\ln\frac{1}{2}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}=-\ln 1+\ln 2=\ln 2$$

③ $y=\ln x$, $y=0$, $x=2$ Method I

$$\text{Area}=\iint_G dA=\int_1^2\int_0^{\ln x} dy\,dx$$

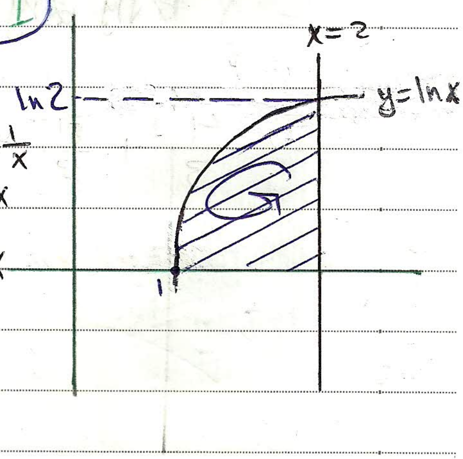
$$=\int_1^2 \ln x\,dx=(x\ln x-x)\Big]_1^2$$

$u=\ln x \rightarrow du=\frac{1}{x}$

$dv=dx \rightarrow v=x$

$x\ln x-\int \frac{\cancel{x}\cdot 1}{\cancel{x}}dx$

$x\ln x-x$



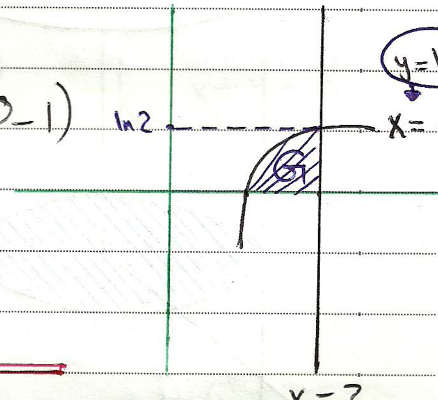
$$2\ln 2-2-(\ln 1-1)=2\ln 2-1$$

Method II

$$\text{Area} = \iint_G dA = \int_0^{\ln 2}\int_{e^y}^{2} dx\,dy = \int_0^{\ln 2} (2 - e^y)\,dy$$

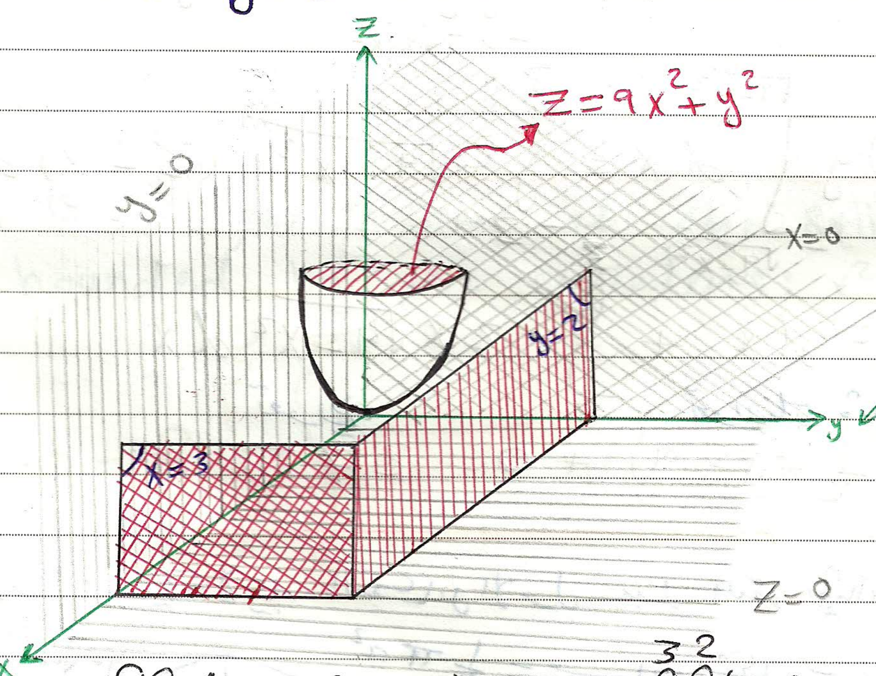
$$= (2y - e^y)\Big]_0^{\ln 2} = 2\ln 2 - 2 - (0 - 1)$$

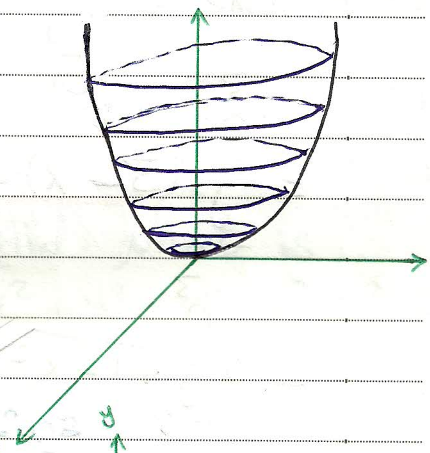
$$= 2\ln 2 - 1$$

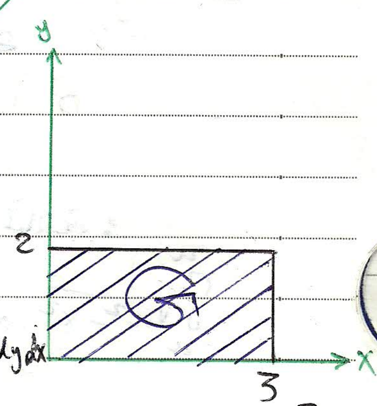


e.g: Find the volume of the Solid:

① That is below the paraboloid $z = 9x^2 + y^2$ and above $z = 0$ and laterally by the planes $x = 0$, $y = 0$, $x = 3$ and $y = 2$.





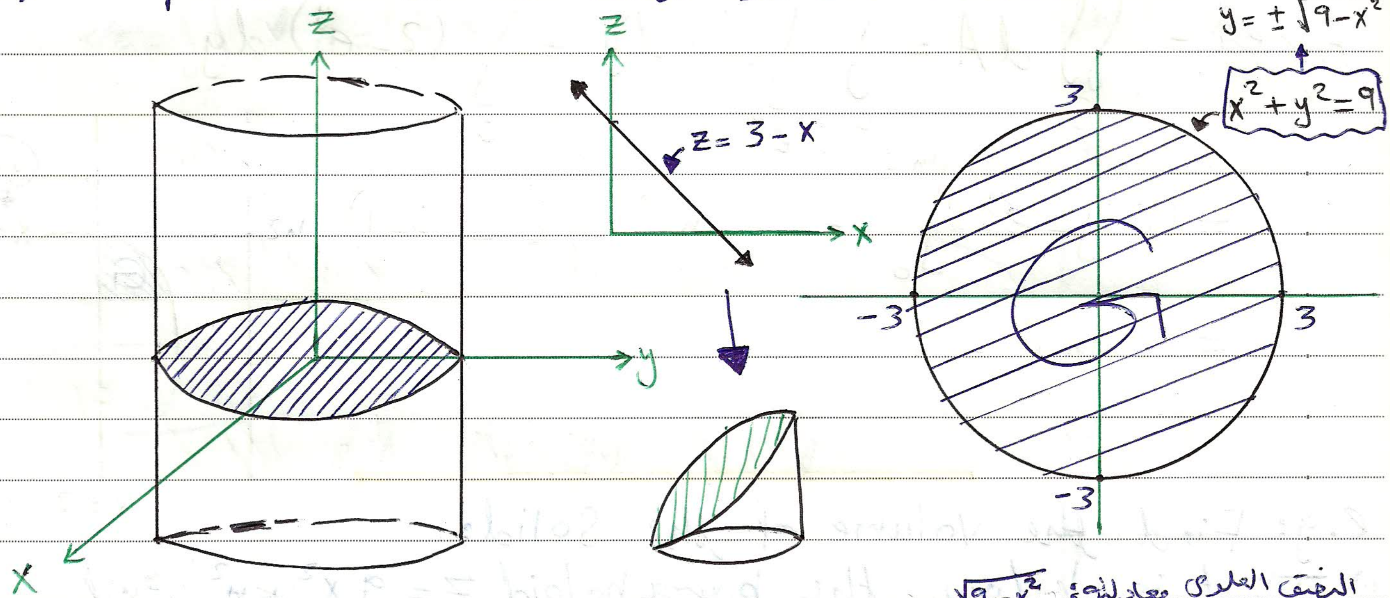


$$\text{Volume} = \iint_G (9x^2 + y^2)\,dA = \int_0^3\int_0^2 (9x^2 + y^2)\,dy\,dx$$

$$= \int_0^3 \left(9x^2 y + \frac{y^3}{3}\right)\Big]_0^2 dx = \int_0^3 18x^2 + \frac{8}{3}\,dx = 6x^3 + \frac{8}{3}x\Big]_0^3$$

$$= 170$$

② That is bounded by the cylinder $x^2+y^2=9$, the planes $z=0$ and $z=3-x$.



الضلع العلوي معادلته: $\sqrt{9-x^2}$

الضلع السفلي معادلته: $-\sqrt{9-x^2}$

$$\text{Volume} = \iint_G (3-x)\, dA = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{+\sqrt{9-x^2}} (3-x)\, dy\, dx$$

$$= \int_{-3}^{3} (3-x)y \Big]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx = \int_{-3}^{3} 2\sqrt{9-x^2}\,(3-x)\, dx$$

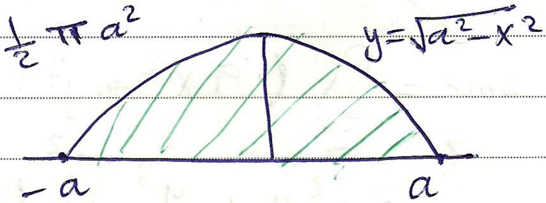
$$= 6\int_{-3}^{3} \sqrt{9-x^2}\, dx + \int_{-3}^{3} -2x\sqrt{9-x^2}\, dx$$

تعويضات مثلثية — تعويض

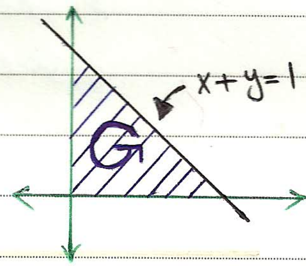
$$= 6\left(\tfrac{1}{2}\times\pi\times 9\right) + 0 = 27\pi$$

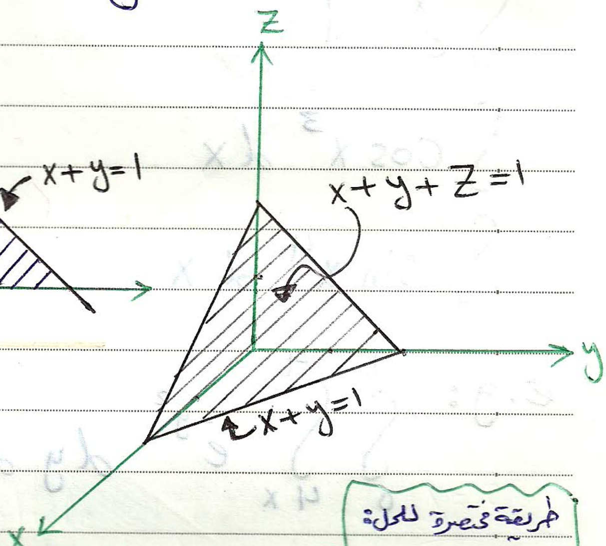
طريقة بسيطة لإيجاد التكامل الاول بدل التعويضات المثلثية:

$$\int_{-a}^{a} \sqrt{a^2-x^2}\, dx = \text{مساحة الشكل} = \tfrac{1}{2}\pi a^2$$



③ That is bounded by the coordinate planes and the plane $x+y+z=1$. (tetera hedron).





$$\text{Volume} = \iint_G (1-x-y)\,dA$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 y - xy - \frac{y^2}{2} \Big]_0^{1-x} dx$$

طريقة مختصرة للحل:

حجم أي جسم ينتهي بنقطة $= \frac{1}{3} *$ مساحة القاعدة $*$ الارتفاع

$$= \int_0^1 (1-x) - (x-x^2) - \frac{(1-x)^2}{2}\,dx = \int_0^1 x^2 - 2x + 1 - \frac{(1-x)^2}{2}\,dx$$

$$= \frac{x^3}{3} - x^2 + x + \frac{(1-x)^3}{6}\Big]_0^1 = \frac{1}{3} - \cancel{1} + \cancel{1} - 0 - \frac{1}{6} = \boxed{\frac{1}{6}}$$

Ex: Find the volume of the solid that is bounded by the cylinders: $x^2+y^2=25$, $x^2+z^2=25$.

$$\text{Volume} = 8\int_0^5 \int_0^{\sqrt{25-x^2}} \sqrt{25-x^2}\;dy\,dx = 8\int_0^5 25-x^2\,dx = \frac{2000}{3}$$

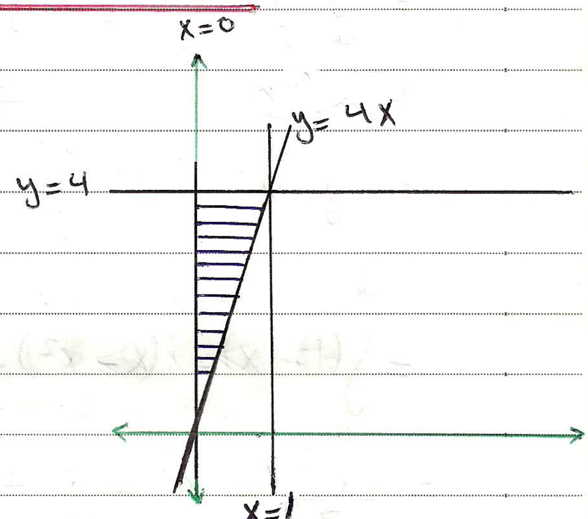
Reversing the order of integration:

$$\int e^{x^2}\,dx$$

$$\int \cos x^3\,dx$$

$$\int \sin x^4\,dx$$

e.g: $\displaystyle\int_0^1\int_{4x}^4 e^{-y^2}\,dy\,dx$



$x=0 \longrightarrow x=1$

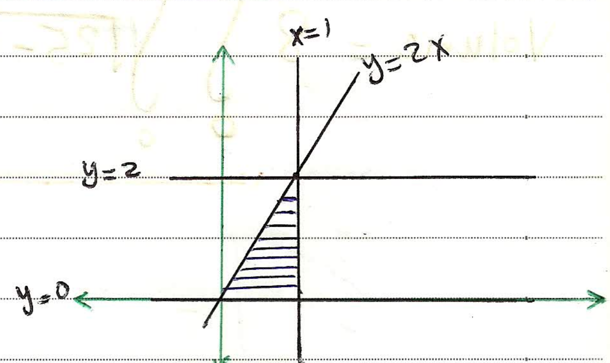
$\boxed{y=4x} \longrightarrow y=4$

$x=\frac{y}{4}$

$$\int_0^4\int_0^{\frac{y}{4}} e^{-y^2}\,dx\,dy=\int_0^4 \frac{y}{4}e^{-y^2}\,dy=\frac{1}{4}*\frac{-1}{2}\left(e^{-y^2}\right)\Big]_0^4$$

$$=\frac{-1}{8}\left(e^{-16}-1\right)=\frac{1}{8}\left(1-e^{-16}\right)$$

e.g: $\displaystyle\int_0^2\int_{\frac{y}{2}}^1 \cos x^2\,dx\,dy$



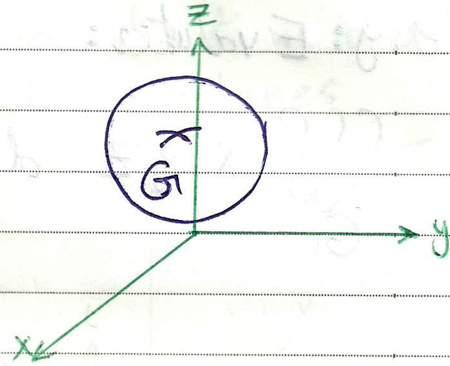
$y=0 \longrightarrow y=2$

$y=2x \leftarrow \boxed{x=\frac{y}{2}} \longrightarrow x=1$

$$\int_0^1\int_0^{2x}\cos x^2\,dy\,dx=\int_0^1 2x\cos x^2\,dx$$

$$=\sin x^2\Big]_0^1=\sin 1-\sin 0=\sin 1$$

* Triple Integral :



G : solid "مجسم"

$$\iiint f(x,y,z)\,dv$$

$$\left.\begin{aligned}
dv &= dz\,dy\,dx \\
dv &= dz\,dx\,dy \\
dv &= dy\,dz\,dx \\
dv &= dx\,dy\,dz \\
dv &= dx\,dz\,dy \\
dv &= dy\,dx\,dz
\end{aligned}\right\}\ \text{6- permutation}$$

* If G is a rectangular box then all the 6 limits are constant.

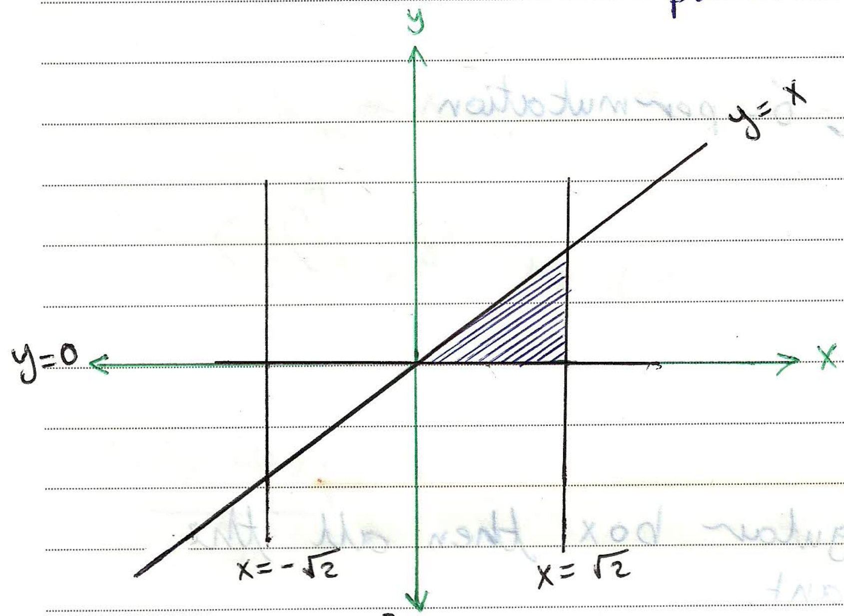
* If G is a closed bounded solid then the Volume of G is given by:

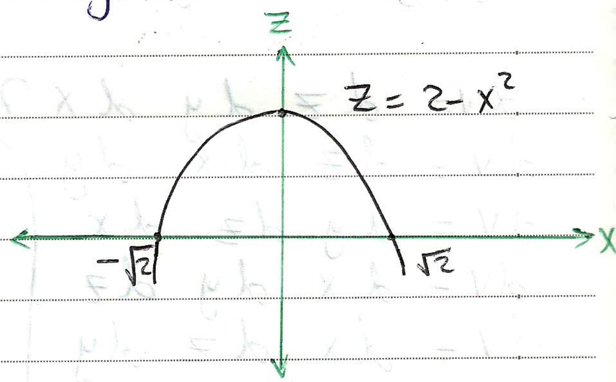
$$\text{Volume} = \iiint_G dv$$

e.g: Evaluate:

$$\iiint_G x\,y\,z\;dv$$

Where G is the solid in the first octant bounded by the planes $y=x$, $y=0$, $z=0$ and the parabolic cylinder $z=2-x^2$





$$0 \le z \le 2-x^2$$
$$0 \le x \le \sqrt{2}$$
$$0 \le y \le x$$

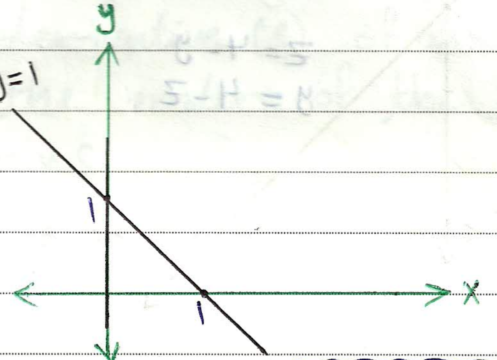
$$\text{Volume} = \int_0^{\sqrt{2}}\int_0^{x}\int_0^{2-x^2} x\,y\,z\;dz\,dy\,dx = \int_0^{\sqrt{2}}\int_0^{x} xy\,\frac{z^2}{2}\Big]_0^{2-x^2} dy\,dx$$

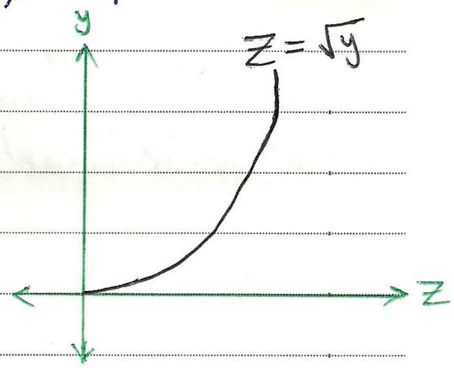
$$=\frac{1}{2}\int_0^{\sqrt{2}}\int_0^{x} xy(2-x^2)\,dy\,dx = \frac{1}{2}\int_0^{\sqrt{2}}\int_0^{x}(2xy-x^3y)\,dy\,dx$$

$$=\frac{1}{2}\int_0^{\sqrt{2}}\left(xy^2-x^3\frac{y^2}{2}\right)\Big]_0^{x} dx = \frac{1}{2}\int_0^{\sqrt{2}} x^3-\frac{x^5}{2}\;dx$$

$$=\frac{1}{2}\left(\frac{x^4}{4}-\frac{x^6}{12}\right)\Big]_0^{\sqrt{2}} = \frac{1}{2}\left(1-\frac{2}{3}\right)=\frac{1}{6}$$

e.g: Find the volume of the solid that is bounded by the surface $z=\sqrt{y}$ and the planes $x+y=1$, $x=0$ and $z=0$.





$0 \leq z \leq \sqrt{y}$
$0 \leq x \leq 1$
$0 \leq y \leq 1-x$

$$\text{Volume} = \int_0^1\int_0^{1-x}\int_0^{\sqrt{y}} dz\,dy\,dx = \boxed{\int_0^1\int_0^{1-x} \sqrt{y}\,dy\,dx}$$

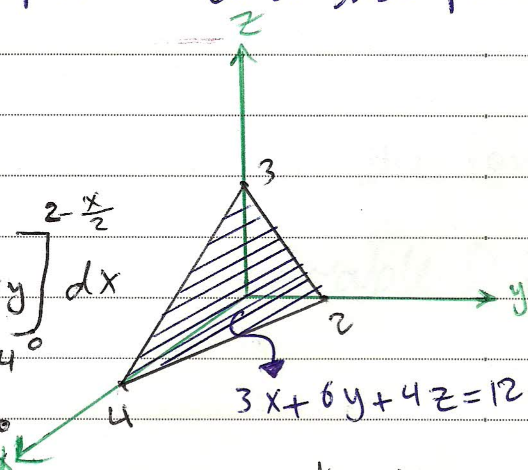
ملاحظة: نفسها الحجم باستخدام double

$$= \int_0^1 \frac{2}{3} y^{3/2}\Big]_0^{1-x} dx = \frac{2}{3}\int_0^1 (1-x)^{3/2}\,dx$$

$$= \frac{2}{3} * \frac{-2}{5} * (1-x)^{5/2}\Big]_0^1 = \frac{4}{15}$$

Ex: Find the volume of the solid that is:

① bounded by the coordinate planes and the plane $3x+6y+4z=12$

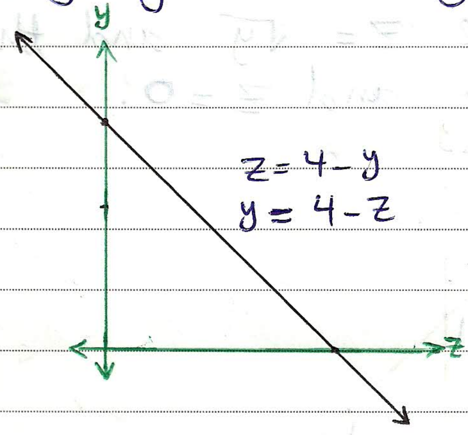


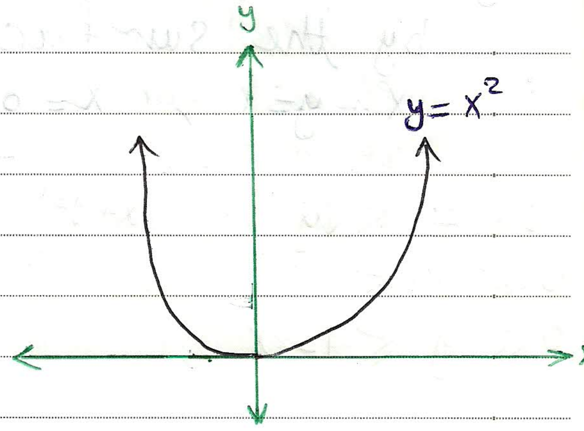
$$\text{Volume} = \int_0^4\int_0^{2-\frac{x}{2}}\int_0^{3-\frac{3}{2}y-\frac{3}{4}x} dz\,dy\,dx$$

$$= \int_0^4\int_0^{2-\frac{x}{2}} 3-\frac{3}{2}y-\frac{3}{4}x\,dy\,dx = \int_0^4 3y-\frac{3}{4}y^2-\frac{3}{4}xy\Big]_0^{2-\frac{x}{2}} dx$$

$$= \int_0^4 \frac{3}{16}x^2+3-\frac{3}{2}x\,dx = \frac{x^3}{16}+3x-\frac{3}{4}x^2\Big]_0^4$$

② bounded by $y = x^2$, $y + z = 4$ and $z = 0$.





③ bounded by $z = x^2 + y^2$ and $y = 2x$.

Chapter 3

30 / 3 / 2009

Differential Equations "المعادلة التفاضلية"

Def : A differential equation (D.E) is an equation that contains derivative (s) of unknown function its order is the order of the highest derivative that it contains.

- To solve a D.E is to find the unknown function that satisfies it.

examples:

① $y'' + xy' - y^2 = e^x$ — D.E of 2^{nd} order. ordinary D.E (O.D.E)

② $y^{(5)} + y^7 = 0$ — (O.D.E) of 5^{th} order.

③ $y^{(iv)} + y''' = e^x$ — (O.D.E) of 4^{th} order.

④ $x\,dy + y\,dx = 0 \longrightarrow x\dfrac{dy}{dx} + y = 0$ — (O.D.E) of 1^{st} order.

⑤ $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial y} = 0$ — D.E of 2^{nd} order. Partial D.E (P.D.E)

Seperable D.E:

Def: Any D.E that can be written in the form:

$f(y)\,dy = g(x)\,dx$ is called seperable D.E

- To solve it, just integrate.

e.g: Solve:

① $y' = 2x$

$\frac{dy}{dx} = 2x \longrightarrow \int dy = \int 2x\,dx \longrightarrow \boxed{y = x^2 + C}$

general solution (G.S) or (g.s)

Condition(s)

D.E + $y(x_0) = y_0$ $\longrightarrow$ Initial value Problem (IVP).

$y'(x_0) = y_1$

$y''(x_0) = y_2$

$\vdots$

② $y' = \cos x$, $y\left(\frac{\pi}{2}\right) = 5$

$\frac{dy}{dx} = \cos x \longrightarrow \int dy = \int \cos x\,dx \longrightarrow y = \sin x + C$ (G.S)

$y\left(\frac{\pi}{2}\right) = 5 \longrightarrow \sin\left(\frac{\pi}{2}\right) + C = 5 \longrightarrow C = 4$

$y = \sin x + 4$ Particular Solution (P.S)

$y = f(x) \longrightarrow$ explicit Solution else its called implicit Solution.

③ $xy\frac{dy}{dx} = 4$

$\int y\,dy = \int \frac{4}{x}\,dx \longrightarrow \frac{y^2}{2} = 4\ln x + C$ G.S (implicit)

$y^2 = 8\ln x + C_1$, $C_1 = 2C$

$y = \pm\sqrt{8\ln x + C_1}$ (explicit)

Chapter 3

30 3 9

Day .... Month .... Year (200 9)

④ $y' = 1 + x + y^2 + xy^2$

$y' = (1+x) + y^2(1+x) \longrightarrow \frac{dy}{dx} = (1+x)(1+y^2)$

$\int \frac{dy}{(1+y^2)} = \int (1+x)\,dx \longrightarrow \tan^{-1} y = x + \frac{x^2}{2} + c$ G.S (implicit)

$y = \tan\left(x + \frac{x^2}{2} + c\right)$ (explicit)

⑤ $y' = e^{x+y}$

$\frac{dy}{dx} = e^x \cdot e^y \longrightarrow \int e^{-y}\,dy = \int e^x\,dx$

$-e^{-y} = e^x + c$ G.S (implicit)

$e^{-y} = -e^x - c \longrightarrow \ln e^{-y} = -\ln e^x - \ln c \longrightarrow -y = -x - \ln c$

$y = x + c_1$, $c_1 = \ln c$ (explicit)

⑥ $\frac{dy}{dx} = 2xy$

$\int \frac{dy}{y} = \int 2x\,dx \longrightarrow \ln y = x^2 + c_1$ G.S (implicit)

$e^{\ln y} = e^{x^2 + c_1} \longrightarrow y = e^{x^2} c$, $c = e^{c_1}$ (explicit)

لا تنسونا من صالح دعائكم

مهندس متشائل

^ _ ^

* Linear D.E:

Def: Any D.E that can be written in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

is called Linear D.E (L.D.E) where $a_0, a_1, \ldots, a_n$ and f are continuous functions.

If $f(x)=0$ then its called homogeneous.
If $a_0, a_1, \ldots, a_n$ are constant functions then its called with constant coefficients.

eg: which of the following D.E is linear?

① $(x^2+1)y'' - xy' + y = x+3$ (L.D.E)
not homogeneous, not with constant coefficients.

② $y''' - y'' + y^2 = 0$ non-Linear

③ $\frac{dy}{dx} = \frac{y}{x+y}$ $y=$ dep. non-Linear

* $\frac{dx}{dy} = \frac{x+y}{y} \longrightarrow \frac{dx}{dy} - \frac{1}{y}x = 1$ Linear, x: dep.

④ $y'' + xy' = 0$ (L.D.E), homogeneous,
not with constant coefficients.

⑤ $\frac{dy}{dx} + y = xy \longrightarrow \frac{dy}{dx} + (1-x)y = 0$ (L.D.E)
homogeneous, not with constant coefficients.

⑥ $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$ (L.D.E), with constant coefficients,
not homogeneous.

⑦ $yy' + y = x$ non-Linear.

⑧ $xy' + xy = x\sin x$, $x>0$ (L.D.E), not homogeneous,
with constant coefficients.

Chapter 3 Day 6 Month 4 Year (2009)

Solving 1st order L.D.E:

$a_1(x)\,y' + a_0(x)\,y = f(x)$

$\boxed{y' + P(x)\,y = g(x)} \rightarrow$ standard form.

* To solve this:

① Find the integrating factor $M(x) = e^{\int p(x)\,dx}$.

② Apply $y = \frac{1}{M(x)}\left[\int M(x)\,g(x)\,dx + C\right]$ to get G.S.

e.g: Solve:

① $y' + 2xy = x$

$p(x) = 2x \quad , \quad g(x) = x$

$M(x) = e^{\int 2x\,dx} = e^{x^2}$

$y = \frac{1}{e^{x^2}}\left[\int x e^{x^2}\,dx + C\right] = e^{-x^2}\left[\frac{1}{2}e^{x^2} + C\right] = \frac{1}{2} + Ce^{-x^2}$ (G.S)

② $y' + y\tan x = \sin 2x \qquad 0 < x < \frac{\pi}{2}$

$y\left(\frac{\pi}{4}\right) = 5$

$M(x) = e^{\int \tan x\,dx} = e^{-\int \frac{-\sin x}{\cos x}} = \frac{1}{\cos x}$

$y = \cos x\left[\int \frac{\sin 2x}{\cos x}\,dx + C\right] = \cos x\left[\int \frac{2\sin x\,\cancel{\cos x}}{\cancel{\cos x}}\,dx + C\right]$

$y = \cos x\left[2\cos x + C\right] = -2\cos^2 x + C\cos x$ (G.S)

$y\left(\frac{\pi}{4}\right) = 5 \longrightarrow -1 + \frac{1}{\sqrt{2}}C = 5 \longrightarrow C = 6\sqrt{2}$

$y = -2\cos^2 x + 6\sqrt{2}\cos x$. (P.S)

③ $x y' + (x+1) y = x^2 \quad , \quad x > 0$

$y' + \left(1 + \frac{1}{x}\right) y = x$

$M(x) = e^{\int (1+\frac{1}{x}) dx} = e^{x + \ln x} = x e^x$

بالاجزاء

$y = \frac{1}{x e^x} \left[\int x^2 e^x \, dx + C \right]$

x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

$y = \frac{1}{x e^x} \left[x^2 e^x - 2 x e^x + 2 e^x + C \right]$ (G.S)

④ $\frac{dy}{dx} = \frac{y}{x+y} \longrightarrow \frac{dx}{dy} = \frac{x+y}{y} \longrightarrow \frac{dx}{dy} - \frac{1}{y} x = 1$

$M(y) = e^{\int -\frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$

$x = y \left[\int \frac{1}{y} dy + C \right] = y \left[\ln y + C \right]$

$x = y \ln y + C y.$ (G.S)

* Bernoulli Equation:

$$\boxed{y' + P(x)\, y = g(x)\, y^n} \rightarrow \text{non-linear} \qquad , n \neq 0 \ , n \neq 1$$

Assume $u = y^{1-n}$

$$\frac{du}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$$

$$(1-n)\, y^{-n} * \boxed{y' + P(x)\, y = g(x)\, y^n} \Rightarrow (1-n)\, y^{-n} y' + (1-n) P(x) y^{1-n} = (1-n)\, g(x)$$

$$\boxed{\frac{du}{dx} + (1-n)\, P(x)\, u = (1-n)\, g(x)} \rightarrow \text{L.D.E}$$

e.g: Solve:

① $y' - 4y = -12 y^2$

$\boxed{n=2}$ let $u = y^{1-2} \rightarrow u = y^{-1}$

$$u' - 4*(-1)\, u = (-12)*(-1) \longrightarrow \boxed{u' + 4u = 12} \rightarrow \text{L.D.E}$$

$$M(x) = e^{\int 4dx} = e^{4x}$$

$$\therefore u = \frac{1}{e^{4x}}\left[\int 12 e^{4x} dx + c\right] \rightarrow u = e^{-4x}\left[3e^{4x} + c\right]$$

$$u = 3 + c\, e^{-4x}$$

$$y^{-1} = 3 + c\, e^{-4x} \longrightarrow y = \frac{1}{3 + c\, e^{-4x}} \quad (G.S)$$

Day 6 Month 4 Year (2009)

② $y' + y = \sqrt{xy}$

$y' + y = x^{\frac{1}{2}} y^{\frac{1}{2}}$

$\boxed{n = \frac{1}{2}}$ let $u = y^{1-\frac{1}{2}} \longrightarrow u = y^{\frac{1}{2}}$

$u' + \frac{1}{2} u = \frac{1}{2} x^{\frac{1}{2}}$

$M(x) = e^{\int \frac{1}{2} dx} = e^{\frac{1}{2}x}$

$u = \frac{1}{e^{\frac{1}{2}x}} \left[\int \frac{1}{2} x^{\frac{1}{2}} e^{\frac{1}{2}x} dx + C \right]$

$u = e^{-\frac{1}{2}x} \left[\int^{x} \frac{1}{2} t^{1/2} e^{\frac{1}{2}t} dt + C \right]$!!! تكامل لا يحل

Ex: ③ $y' - \sigma y = -\epsilon y^2$ (σ, ϵ) positive constants.

$\boxed{n = 2}$ let $u = y^{1-2} \longrightarrow u = y^{-1}$

$u' - (\sigma) * (-1) u = (-\epsilon)(-1) \longrightarrow u' + \sigma u = \epsilon$

$M(x) = e^{\int \sigma dx} = e^{\sigma x}$

$\therefore u = \frac{1}{e^{\sigma x}} \left[\int \epsilon e^{\sigma x} dx + c \right] \longrightarrow u = e^{-\sigma x} \left[\frac{\epsilon}{\sigma} e^{\sigma x} + c \right]$

$u = \frac{\epsilon}{\sigma} + c e^{-\sigma x}$

$y^{-1} = \frac{\epsilon}{\sigma} + C e^{-\sigma x} \longrightarrow y = \frac{1}{\frac{\epsilon}{\sigma} + C e^{-\sigma x}}$ (G.S)

$\sqrt{-4} = \sqrt{4}\,\sqrt{-1} = 2i$

Def: $i = \sqrt{-1}$, $i^2 = -1$

$Z = x + iy$; $x, y \in \mathbb{R}$

Complex number.

$Re(Z) = x$ real

$Im(Z) = y$ imaginary

$\overline{Z} = x - iy$ • conjugate of Z

examples:

$\overline{3+2i} = 3 - 2i$

$\overline{-5i} = 5i$

$\overline{7} = 7$

$|Z| = \sqrt{x^2 + y^2} = r$ modulus of Z

$Z\overline{Z} = (x+iy)(x-iy)$

$= x^2 - i^2 y^2$

$= x^2 + y^2$

$= |Z|^2$

$e^{i\theta} = \cos\theta + i\sin\theta$ Euler relation.

$e^{-i\theta} = \cos\theta - i\sin\theta$

e.g: Solve:

$$Z^2 + 2Z + 5 = 0$$

$$Z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$Z = -1 \pm 2i$$

If $Z_1 = x_1 + y_1 i$ is a root of

$a_n Z^n + a_{n-1} Z^{n-1} + \dots + a_0 = 0$, $a_0, a_1, \dots, a_n$ are real numbers.

Then: $\overline{Z}_1 = x_1 - i y_1$ is a root.

2^{nd} order L.D.E'S homo with constant coeficients:

$$\boxed{a y'' + b y' + C y = 0}$$

- To solve this equation:

① write auxiliary (characteristic) equation.

$\boxed{a m^2 + b m + C = 0}$

② Solve this equation. (m_1 , m_2 are the roots).

③ The (G.S) will be:

$$y = \begin{cases} C_1 e^{m_1 x} + C_2 e^{m_2 x} & , m_1, m_2 \in R, \ m_1 \neq m_2 \\ C_1 e^{m_1 x} + C_2 x e^{m_1 x} & , m_1 = m_2 \in R \\ e^{\alpha x} \left[C_1 \cos \beta x + C_2 \sin \beta x \right] & , m_{1,2} = \alpha \pm \beta i \end{cases}$$

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e.g: Solve:

① $y'' - 3y' + 2y = 0$

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m_1 = 1 \; , \; m_2 = 2$ distict roots

$\therefore y = C_1 e^{x} + C_2 e^{2x}$ (G.S)

② $y'' - 6y' + 9y = 0$

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m_1 = 3 \; , \; m_2 = 3$ repeated roots

$\therefore y = C_1 e^{3x} + C_2 x e^{3x}$ (G.S)

③ $y'' + 2y' + 5y = 0$

$m^2 + 2m + 5 = 0$

$m = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2} \Rightarrow m_{1,2} = -1 \pm 2i$

$y = e^{-x}[C_1 \cos 2x + C_2 \sin 2x]$ (G.S)

D.E + Conditions $\longrightarrow$ B V P

at different values of x

Boundary value problem.

"حدية"

④ $y'' + y = 0$, $y(0) = 4$
(BVP) $y(\frac{\pi}{2}) = -1$

$m^2 + 1 = 0$
$m^2 = -1$
$m_{1,2} = \pm i$
$y = e^{0x}[C_1 \cos x + C_2 \sin x]$

$\therefore y = C_1 \cos x + C_2 \sin x$ (G.S)

$y(0) = 4 \rightarrow C_1 = 4$
$y(\frac{\pi}{2}) = -1 \rightarrow C_2 = -1$
$\therefore y = 4\cos x - \sin x$ (P.S)

⑤ $y'' - 2y' = 0$, $y(0) = 4$
(IVP) $y'(0) = 6$

$m^2 - 2m = 0$
$m(m-2) = 0$
$m_1 = 0$, $m_2 = 2$
$y = C_1 + C_2 e^{2x}$ (G.S)
$y' = 2C_2 e^{2x}$
$y(0) = 4 \rightarrow C_1 + C_2 = 4 \longrightarrow \circledast$
$y'(0) = 6 \rightarrow 2C_2 = 6 \longrightarrow C_2 = 3$
$\therefore C_1 = 1$
$\therefore y = 1 + 3e^{2x}$ (P.S)

* m_1, m_2 : $(m - m_1)(m - m_2) = 0$ * للحل بالعكس *
$m^2 - (m_1 + m_2)m + m_1 m_2 = 0$

e.g: write the D.E whose solution:

① $y = C_1 e^{2x} + C_2 e^{-3x}$

$m_1 = 2 \ , \ m_2 = -3$

$m^2 + m - 6 = 0$

$y'' + y' - 6y = 0$

② $y = C_1 e^{3x} + C_2 \, x \, e^{3x}$

$m_1 = m_2 = 3$

$m^2 - 6m + 9 = 0$

$y'' - 6y' + 9y = 0$

③ $y = e^{4x} \left[C_1 \cos 2x + C_2 \sin 2x \right]$

$m_1 = 4 + 2i$

$m_2 = 4 - 2i$

$m^2 - 8m + 20 = 0$

$y'' - 8y' + 20y = 0$

Ex: ④ $y = C_1 \cosh 3x + C_2 \sinh 3x$

$y = C_1 e^{3x} + C_2 e^{-3x}$

$m_1 = 3 \ , \ m_2 = -3$

$m^2 - 9 = 0$

$y'' - 9y = 0$

$y'' - a^2 y = 0$

$y = C_1 e^{ax} + C_2 e^{-ax}$

$y = C_1 \cosh ax + C_2 \sinh ax$

e.g: ① $y''' + 3y'' + 3y' + y = 0$

$m^3 + 3m^2 + 3m + 1 = 0$

$m = -1 \rightarrow -1 + 3 - 3 + 1 = 0$

$(m+1)$ is afactor

	1	3	3	1
$\boxed{-1}$	↓	-1	-2	-1
	1	2	1	"0"

remainder "الباقي"

$(m+1)(m^2 + 2m + 1) = 0$

$(m+1)(m+1)(m+1) = 0$

$m_1 = -1 \;,\; m_2 = -1 \;,\; m_3 = -1$

$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$ (G.S)

② $y''' - 4y' = 0$

$m^3 - 4m = 0$

$m(m^2 - 4) = 0$

$m(m-2)(m+2) = 0$

$m_1 = 0 \;,\; m_2 = 2 \;,\; m_3 = -2$

$y = C_1 e^{0x} + C_2 e^{2x} + C_3 e^{-2x}$

$y = C_1 + C_2 e^{2x} + C_3 e^{-2x}$ (G. S)

③ $y^{iv} + 13y'' + 36y = 0$

$m^4 + 13m^2 + 36 = 0$

$(m^2+4)(m^2+9) = 0$

$m_{1,2} = \pm 2i \quad , \quad m_{3,4} = \pm 3i$

$y = e^{0x}[C_1 \cos 2x + C_2 \sin 2x] + e^{0x}[C_3 \cos 3x + C_4 \sin 3x]$

$y = C_1 \cos 2x + C_2 \sin 2x + C_3 \cos 3x + C_4 \sin 3x$. (G.S)

④ $y^{iv} + 2y^2 + y = 0$

$m^4 + 2m^2 + 1 = 0$

$(m^2+1)(m^2+1) = 0$

$m_{1,2} = \pm i \quad , \quad m_{3,4} = \pm i$

$\therefore y = e^{0x}[C_1 \cos x + C_2 \sin x] + x e^{0x}[C_3 \cos x + C_4 \sin x]$

$y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$. (G.S)

Ex: ⑤ $y^{(4)} + 16y = 0$

$m^4 + 16 = 0$

$m^2 = \sqrt{-16}$ →

Non-Homogeneous L.D.Es :

① Method of undetermined coefficients

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = f(x)$ $a_0, a_1, \dots, a_n$ are constants.

- To solve this equation:

$\boxed{1}$ Solve the corresponding homo equation its solution is denoted by y_h or y_c. (c: Complementary)

$\boxed{2}$ Find the particular solution using the following table:

	$f(x)$	y_p
1.	$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ $(3x^2 - 1 \,/\, \frac{1}{2}x^2 \,/\, x^2 + x + 10d)$ $(x \,/\, -\frac{1}{2}x + 1)$ (7)	$A_n x^n + A_{n-1} x^{n-1} + \dots + A_0$ $Ax^2 + Bx + C$ $Ax + B$ A
2.	$P_n(x)\, e^{\alpha x}$ $(x e^{4x})$ $(-e^{3x})$	$(A_n x^n + \dots + A_0) e^{\alpha x}$ $(Ax + B) e^{4x}$ $A e^{3x}$
3.	$P_n(x) \cos\beta x$ or $Q_n(x) \sin\beta x$ or their sum. $(x\cos 2x \,/\, (3x-1)\sin 2x \,/\, (2x+1)\cos 2x - x\sin 2x$	$(A_n x^n + \dots + A_0)\cos\beta x + (B_n x^n + \dots + B_0)\sin\beta x$ $(Ax+B)\cos 2x + (Cx+D)\sin 2x$
4.	$P_n(x) e^{\alpha x} \cos\beta x$ or $Q_n(x) e^{\alpha x} \sin\beta x$ or their sum. $(e^{4x} \sin 3x)$	$(A_n x^n + \dots + A_0) e^{\alpha x} \cos\beta x + (B_n x^n + \dots + B_0) e^{\alpha x} \sin\beta x$ $Ae^{4x}\cos 3x + Be^{4x}\sin 3x$

* You may need to multiply the assumption of y_p by $x, x^2, \dots$
* The (G.S) : $y = y_h + y_c$.

e.g: Solve:

① $y'' - y' - 2y = x^2$

$m^2 - m - 2 = 0$

$(m+1)(m-2) = 0$

$m_1 = -1 , m_2 = 2$

$\therefore y_h = C_1 e^{-x} + C_2 e^{2x}$.

let $y_p = Ax^2 + Bx + C$.

$y'_p = 2Ax + B$

$y''_p = 2A$

Sub. in ①

$2A - 2Ax - B - 2Ax^2 - 2Bx - 2C = x^2$

$-2Ax^2 + (-2A - 2B)x + (2A - B - 2C) = x^2$

$-2A = 1 \longrightarrow \boxed{A = \frac{-1}{2}}$

$-2A - 2B = 0 \longrightarrow 1 - 2B = 0 \longrightarrow \boxed{B = \frac{1}{2}}$

$2A - B - 2C = 0 \longrightarrow -1 - \frac{1}{2} - 2C = 0 \longrightarrow \boxed{C = \frac{-3}{4}}$

$\therefore y_p = \frac{-1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

$$\boxed{\therefore y = C_1 e^{-x} + C_2 e^{2x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}}$$ (G.S)

② $y''' - y' = 12e^{x}$

$m^3 - m = 0$

$m(m^2 - 1) = 0$

$m(m-1)(m+1) = 0$

$m_1 = 0 \; , \; m_2 = 1 \; , \; m_3 = -1$

$\therefore y_h = C_1 + C_2 e^{x} + C_3 e^{-x}$

let $y_p = (Ae^{x})$ ← مرفوض لانه متكرر $\longrightarrow y_p = Axe^{x}$

$y'_p = Ae^{x} + Axe^{x}$

$y''_p = Ae^{x} + Ae^{x} + Axe^{x} = Axe^{x} + 2Ae^{x}$

$y'''_p = Axe^{x} + Ae^{x} + 2Ae^{x} = Axe^{x} + 3Ae^{x}$

Sub. in the D.E:

$\cancel{Axe^{x}} + 3Ae^{x} - Ae^{x} - \cancel{Axe^{x}} = 12e^{x}$

$2Ae^{x} = 12e^{x} \longrightarrow 2A = 12 \longrightarrow \boxed{A = 6}$

$\therefore y_p = 6xe^{x}$

$\boxed{\therefore y = C_1 + C_2 e^{x} + C_3 e^{-x} + 6xe^{x}}$ (G.S)

③ $y'' - 2y' - 3y = \sin x$

$m^2 - 2m - 3 = 0$

$(m+1)(m-3) = 0$

$m_1 = -1 \quad , m_2 = 3$

$\therefore y_c = C_1 e^{-x} + C_2 e^{3x}$

let $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

Sub. in the D.E

$-A\cos x - B\sin x + 2A\sin x - 2B\cos x - 3A\cos x - 3B\sin x = \sin x$

$(-A - 2B - 3A)\cos x + (-B + 2A - 3B)\sin x = \sin x$

$-4A - 2B = 0 \longrightarrow -2A - B = 0 \dashrightarrow ①$

$-4B + 2A = 1 \longrightarrow ②$

$① + ② \Rightarrow -5B = 1 \rightarrow B = -0.2 \longrightarrow A = -\frac{1}{2}B \rightarrow A = 0.1$

$\therefore y_p = 0.1\cos x - 0.2\sin x$

$$\boxed{\therefore y = C_1 e^{-x} + C_2 e^{3x} + 0.1\cos x - 0.2\sin x} \quad (G.S)$$

④ $y'' - 2y' = x + e^{3x}$

$m^2 - 2m = 0 \longrightarrow m(m-2) = 0 \rightarrow m_1 = 0 \ , m_2 = 2$

$\therefore y_h = C_1 + C_2 e^{2x}$

let $y_p = \cancel{Ax + B + Ae^{3x}} \longrightarrow y_p = Ax^2 + Bx + Ae^{3x}$

$y_p' = 2Ax + B + 3Ae^{3x}$. $\quad y_p'' = 2A + 9Ae^{3x}$

Sub. in the D.E : $2A + 9Ae^{3x} - 4Ax - 2B - 6Ae^{3x} = x + e^{3x}$

$3Ae^{3x} - 4Ax + 2A - 2B = x + e^{3x} \rightarrow 2A - 2B = 0 \rightarrow A = B$

* e.g: write the true assumption for y_p:

① $y'' + y = 7\cos x - \sin x + \sin 4x$

$m^2 + 1 = 0 \longrightarrow m_{1,2} = \pm i$

$y_h = C_1 \cos x + C_2 \sin x$

$y_p = Ax\cos x + Bx\sin x + C\cos 4x + D\sin 4x$

② $y''' + y' = x\cos x + 1$

$m^3 + m = 0 \longrightarrow m(m^2+1) = 0$

$m_1 = 0,\ m_{2,3} = \pm i$

$y_h = C_1 + C_2\cos x + C_3 \sin x$

$y_p = (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x + Ex$

③ $y'' - y' = 2\sinh x$

$y'' - y' = e^x - e^{-x}$

$m^2 - m = 0 \longrightarrow m(m-1) = 0 \longrightarrow m_1 = 0,\ m_2 = 1$

$y_h = C_1 + C_2 e^x$

$y_p = Axe^x + Be^{-x}$

② Variation of parameters method:

Review:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = -2 - 3 = -5$$

$$\begin{bmatrix} + & - & + & \cdot & \cdot & \cdot \\ - & + & - & \cdot & \cdot & \cdot \\ + & - & + & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ -2 & 3 & 5 \end{bmatrix}$$

$$\det A = (2)(20-3) - (-1)(0+2) + (3)(0+8) = 34 + 2 + 24 = 60$$

* Cramer's rule:

$2x+1=5$
$x-y=1$

$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = -2-1=-3 \neq 0$

$x = \dfrac{\begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}}{-3} = \dfrac{-5-1}{-3} = \dfrac{-6}{-3} = 2$

$y = \dfrac{\begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}}{-3} = \dfrac{2-5}{-3} = \dfrac{-3}{-3} = 1$

Def: The Wrouskian of $y_1, y_2, \ldots, y_n$ is

$$W(y_1, y_2, \ldots, y_n) = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

- if $W(y_1, y_2, \ldots, y_n) \neq 0$ then $y_1, y_2, \ldots, y_n$ are called Linearly independent otherwise, they are called Linearly dependent.

Note:
If $y_1, y_2, \dots, y_n$ Linearly independent solutions of a homo. L.D.E. Then the set $\{y_1, y_2, \dots, y_n\}$ is called a Fundamental set of solutions.

- To solve $a_n(x)\, y^{(n)} + \dots + a_0(x)\, y = g(x)$:

① Solve the corresponding homo equation
y_h or y_c $= C_1 y_1 + C_2 y_2 + \dots + C_n y_n$

② Assume $y_p = u_1(x)\, y_1 + u_2(x)\, y_2 + \dots + u_n(x)\, y_n$

$$u_1' = \frac{\begin{bmatrix} 0 & y_2 \cdots & y_n \\ 0 & y_2' & y_n' \\ f(x) & y_2 & y_n^{(n-1)} \end{bmatrix}}{W(y_1, y_2, \dots, y_n)} \longrightarrow u_1 = \int u_1' \quad , f(x) = \frac{g(x)}{a_n(x)}$$

$$u_n' = \frac{\begin{bmatrix} y_1 & \dots & 0 \\ y_1' & \dots & 0 \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & \dots & f(x) \end{bmatrix}}{W(y_1, y_2, \dots, y_n)} \longrightarrow u_n = \int u_n'$$

③ (G.S) : $y = y_h + y_p$

* Note: 2^{nd} order

$$y_p = u_1 y_1 + u_2 y_2 = y_1 \int \frac{\begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix}}{W} dx + y_2 \int \frac{\begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix}}{W}$$

$$\boxed{y_p = -y_1 \int \frac{f(x)\, y_2}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx}$$

يمكن حفظها كقانون

e.g: Solve :

① $y'' + y = \tan x \quad , \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

$y'' + y = 0 \longrightarrow m^2 + 1 = 0 \longrightarrow m_{1,2} = \pm i$

$y_c = C_1 \cos x + C_2 \sin x$

$y_1 = \cos x \quad , \quad y_2 = \sin x \quad , \quad f(x) = \tan x$

$y_p = u_1 y_1 + u_2 y_2$

$$W(y_1, y_2) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$u_1' = \frac{\begin{bmatrix} 0 & \sin x \\ \tan x & \cos x \end{bmatrix}}{1} = -\sin x \tan x$$

$$u_1 = \int -\sin x \tan x \, dx = \int -\sin x \frac{\sin x}{\cos x} dx = \int \frac{\cos^2 x - 1}{\cos x} dx$$

$$u_1 = \int (\cos x - \sec x) \, dx = \sin x - \ln(\sec x + \tan x)$$

$$u_2' = \frac{\begin{bmatrix} \cos x & 0 \\ -\sin x & \tan x \end{bmatrix}}{1} = \cos x \tan x = \sin x$$

$$u_2 = \int \sin x \, dx = -\cos x$$

$$\therefore y_p = \cancel{\sin x \cos x} - \cos x \ln(\sec x + \tan x) - \cancel{\sin x \cos x}$$

$$\therefore y_p = -\cos x \ln(\sec x + \tan x)$$

$$\therefore y = C_1 \cos x + C_2 \sin x - \cos x \ln(\sec x + \tan x) \quad (G.S)$$

② $y'' + 4y' + 4y = x^{-2} e^{-2x} \quad , \quad x > 0$

$y'' + 4y' + 4y = 0 \longrightarrow m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0 \longrightarrow m_1 = m_2 = -2$

$\therefore y_c = C_1 e^{-2x} + C_2 x e^{-2x}$

$y_1 = e^{-2x} \quad , \quad y_2 = x e^{-2x} \quad , \quad f(x) = x^{-2} e^{-2x}$

let $y_p = u_1 y_1 + u_2 y_2$

$$W(y_1, y_2) = \begin{bmatrix} e^{-2x} & x e^{-2x} \\ -2 e^{-2x} & -2x e^{-2x} + e^{-2x} \end{bmatrix} = e^{-2x} e^{-2x} \begin{bmatrix} 1 & x \\ -2 & -2x+1 \end{bmatrix}$$

$$W(y_1, y_2) = e^{-4x} [\cancel{-2x} + 1 + \cancel{2x}] = e^{-4x} \neq 0$$

$$u_1' = \frac{\begin{bmatrix} 0 & x e^{-2x} \\ x^{-2} e^{-2x} & -2x e^{-2x} + e^{-2x} \end{bmatrix}}{e^{-4x}} = \frac{-x^{-1} e^{-4x}}{e^{-4x}} = -x^{-1}$$

$u_1 = -\ln x$

$$u_2' = \frac{\begin{bmatrix} e^{-2x} & 0 \\ -2 e^{-2x} & x^{-2} e^{-2x} \end{bmatrix}}{e^{-4x}} = \frac{x^{-2} e^{-4x}}{e^{-4x}} = x^{-2}$$

$u_2 = -x^{-1}$

$\therefore y_p = - e^{-2x} \ln x - e^{-2x}$

ممكن اهماله لوجود حد مثله في y_c

$\therefore y = \underline{C_1 e^{-2x}} + C_2 x e^{-2x} - e^{-2x} \ln x - \underline{e^{-2x}} \quad (G.S)$

$= C_1^* e^{-2x} + C_2 x e^{-2x} - e^{-2x} \ln x$

$\hookrightarrow (C_1 - 1)$

③ $y''' + y' = \tan x \quad , \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$y''' + y' = 0 \longrightarrow m^3 + m = 0 \longrightarrow m(m^2+1) = 0$

$m_1 = 0 \; , \; m_{2,3} = \pm i$

$\therefore y = C_1 + C_2 \cos x + C_3 \sin x$

$y_1 = 1 \; , \; y_2 = \cos x \; , \; y_3 = \sin x \; , \; f(x) = \tan x$

let $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$W(y_1, y_2, y_3) = \begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} = \sin^2 x + \cos^2 x = 1$$

$$u_1' = \frac{\begin{bmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{bmatrix}}{1} = \tan x \left(\cos^2 x + \sin^2 x\right) = \tan x$$

$$u_1 = \int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx = -\ln \cos x$$

$$u_2' = \frac{\begin{bmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{bmatrix}}{1} = -\cos x \tan x = -\sin x$$

$u_2 = \cos x$

$$u_3' = \frac{\begin{bmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{bmatrix}}{1} = -\sin x \tan x = \frac{-\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x}$$

$$u_3 = \int (\cos x - \sec x) \, dx = \sin x - \ln(\sec x + \tan x)$$

$\therefore y_p = -\ln(\cos x) + \cos^2 x + \sin^2 x - \sin x \ln(\sec x + \tan x)$

$\therefore y_p = 1 - \ln(\cos x) - \sin x \ln(\sec x + \tan x)$

$\therefore y = C_1 + C_2 \cos x + C_3 \sin x + 1 - \ln(\cos x) - \sin x \ln(\sec x + \tan x)$ (G.S)

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Ex: $\boxed{1}$ Solve:

① $y'' - 5y' + 6y = 2e^{x}$

② $y'' + 9y = 9\sec^2 3x \quad , \; 0 < x < \frac{\pi}{2}$

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③ $y'' + 4y = 3\csc 2x \quad , \quad 0 < x < \frac{\pi}{2}$

$\boxed{2}$ If x, x^2 and $\frac{1}{x}$ are solutions of the homo D.E corresponding to
$$x^3 y''' + x^2 y'' - 2xy' + 2y = 2x^4 \quad , \quad x > 0$$
Determine a particular solutions.

* Power Series Solution:

$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0$, a_0, a_1, a_2 are polynomials

if $a_2(x_0) = 0$ then x_0 is called a singular point.
if $a_2(x_0) \neq 0$ then x_0 is called an ordinary point.

e.g: Find the ordinary and singular points of:

$(x^2 - x)\,y'' - 2x\,y' - y = 0$

$x^2 - x = 0 \longrightarrow x(x-1) = 0 \longrightarrow x = 0$ or $x = 1$ singular pts.
any other point is an ordinary point.

*Recall:

① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$

② $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)}}{(2k+1)!}$

③ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$

$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k)}}{(2k)!}$

④ $\frac{1}{1-x} = 1 + x + x^2 + \ldots\ldots$

$= \sum_{k=0}^{\infty} x^k$, $|x| < 1$

① $\sum_{K=m}^{n} a = (n-m+1)\, a$

$\sum_{K=2}^{4} 5 = 5+5+5 = (4-2+1)\,5$

② $\sum_{K=m}^{n} f(x) = \sum_{K=m+S}^{n+S} f(K-S) = \sum_{K=m-S}^{n-S} f(K+S)$

$X_0 = 0$ is an ordinary

$$y = \sum_{K=0}^{\infty} a_K X^K = a_0 + a_1 X + a_2 X^2 + \dots$$

$$y' = \sum_{K=1}^{\infty} K\, a_K\, X^{(K-1)}$$

$$y'' = \sum_{K=2}^{\infty} K\,(K-1)\, a_K\, X^{(K-2)}$$

⋮

e.g: Solve: $y' - y = 0$ by series:

let $y = \sum_{K=0}^{\infty} a_K X^K$

$$y' = \sum_{K=1}^{\infty} K\, a_K\, X^{(K-1)}$$

Sub. in the D.E:

$$\sum_{K=1}^{\infty} K\, a_K\, X^{(K-1)} - \sum_{K=0}^{\infty} a_K X^K = 0$$

$$\sum_{K=0}^{\infty} (K+1)\, a_{K+1} X^K - \sum_{K=0}^{\infty} a_K X^K = 0 \longrightarrow \sum_{K=0}^{\infty} \{(K+1)\, a_{K+1} - a_K\} X^K = 0$$

$$(k+1)\,a_{k+1} - a_k = 0 \quad , k = 0, 1, 2, 3, \dots$$

$$\boxed{a_{k+1} = \frac{a_k}{k+1} \quad , k = 0, 1, 2, 3, \dots} \rightarrow \text{recurrence relation}$$

let a_0 be afixed real number:

$$a_1 = \frac{a_0}{1}$$

$$a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}$$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{(3)(2)} = \frac{a_0}{3!}$$

$$a_4 = \frac{a_3}{4} = \frac{a_0}{(4)(3)(2)} = \frac{a_0}{4!}$$

$\vdots$

$$\therefore y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \dots$$

$$y = a_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right]$$

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^k}{k!} \longrightarrow \boxed{y = a_0 e^x} \quad \text{(G.S)}$$

$$y' - y = 0 \longrightarrow y' = y \longrightarrow \frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \int dx \longrightarrow \ln y = x + c_1$$

$$\therefore y = C e^x \quad \text{(G.S)}$$

e.g: Solve by Series $y'' + y = 0$

let $y = \sum_{k=0}^{\infty} a_k x^k \longrightarrow y' = \sum_{k=1}^{\infty} k a_k x^{(k-1)} \longrightarrow y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{(k-2)}$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{(k-2)} + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} \{(k+2)(k+1) a_{k+2} + a_k\} x^k = 0$$

$$(k+2)(k+1) a_{k+2} + a_k = 0 \quad , k = 0, 1, 2, 3, \ldots$$

$$\boxed{a_{k+2} = \frac{-a_k}{(k+2)(k+1)} \quad , k = 0, 1, 2, 3, \ldots}$$

let a_0, a_1 be two fixed real numbers:

$a_2 = \frac{-a_0}{(2)(1)}$, $a_3 = \frac{-a_1}{(3)(2)} = \frac{-a_1}{3!}$

$a_4 = \frac{-a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)(1)} = \frac{a_0}{4!}$, $a_5 = \frac{-a_3}{(5)(4)} = \frac{a_1}{5!}$

$a_6 = \frac{-a_4}{(6)(5)} = \frac{-a_0}{6!}$, $a_7 = \frac{-a_1}{7!}$

$\vdots$ $\vdots$

$$\therefore y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 - \frac{a_1}{5!} x^5 + \ldots$$

$$y = a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right]$$

$$y = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)}}{(2k+1)!}$$

$$\boxed{y = a_0 \cos x + a_1 \sin x} \quad (G.S)$$

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Ex: $y'' - y = 0$

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e.g: Solve : $y'' - xy' - y = 0$

let $y = \sum_{k=0}^{\infty} a_k x^k \longrightarrow y' = \sum_{k=1}^{\infty} k a_k x^{(k-1)} \longrightarrow y'' \sum_{k=2}^{\infty} k(k-1) a_k x^{(k-2)}$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{(k-2)} - x \sum_{k=1}^{\infty} k a_k x^{(k-1)} - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} \{(k+2)(k+1) a_{k+2} - k a_k - a_k\} x^k = 0$$

$$(k+2)(k+1) a_{k+2} - k a_k - a_k = 0 \quad , \; k = 0,1,2,3,\ldots$$

$$a_{k+2} = \frac{a_k \cancel{(k+1)}}{(k+2)\cancel{(k+1)}} \longrightarrow \boxed{a_{k+2} = \frac{a_k}{k+2} \; , \; k = 0,1,2,3,\ldots}$$

let a_0, a_1 be two fixed real numbers:

$a_2 = \frac{a_0}{2}$, $a_3 = \frac{a_1}{3}$

$a_4 = \frac{a_2}{4} = \frac{a_0}{(4)(2)}$, $a_5 = \frac{a_3}{5} = \frac{a_1}{(5)(3)}$

$a_6 = \frac{a_0}{(6)(4)(2)}$, $a_7 = \frac{a_1}{(7)(5)(3)}$

$\vdots$ $\qquad\qquad\qquad\qquad\vdots$

$$\therefore y = a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \frac{a_0}{(4)(2)} x^4 + \ldots$$

$$y = a_0 \left[1 + \frac{x^2}{2} + \frac{x^4}{(4)(2)} + \frac{x^6}{(6)(4)(2)} + \ldots\right] + a_1 \left[x + \frac{x^3}{3} + \frac{x^5}{(5)(3)} + \ldots\right]$$

$$\boxed{y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{2^k k! \; x^{(2k+1)}}{(2k+1)!}} \quad (G.S)$$

$(2)(4) \ldots (2k) = 2^k k!$

$(1)(3)(5) \ldots (2k+1) = \frac{(2k+1)!}{2^k k!}$

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*Solving Systems of 1^{st} order L.D.E's:

constants

$$x_1'(t) = a_{11}\, x_1(t) + a_{12}\, x_2(t)$$
$$x_2'(t) = a_{21}\, x_1(t) + a_{22}\, x_2(t)$$

$$x' = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

the above system can be written using matrices as : $x' = Ax$

$$x'(t) = 2\,x(t) + 5\,y(t)$$
$$y'(t) = -x(t) + 2\,y(t)$$

$$x' = \begin{pmatrix} 2 & 5 \\ -1 & 2 \end{pmatrix} x, \quad x = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad x'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

Introduction:

If A is 2×2 - matrix

constant

$Ax = \lambda x$, x - non-zero vector.

eigen value — eigen vector

$|A - \lambda I| = 0 \longrightarrow$ eigen values.

$|A - \lambda I|\, x = \vec{0} \longrightarrow$ eigen vector.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

---

e.g: Find the eigen values and the corresponding eigen vectors of:

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \qquad -\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \longrightarrow (3-\lambda)(-2-\lambda) + 4 = 0$$

$$\longrightarrow -6 + 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\longrightarrow \boxed{\lambda^2 - \lambda - 2 = 0} \text{ characteristic eq.}$$

$(A - \lambda I)x = 0$

$(\lambda+1)(\lambda-2) = 0 \rightarrow \lambda_1 = -1 \ , \ \lambda_2 = 2$ eigen values.

$\lambda_1 = -1 \ , \ X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{aligned} 4x_1 - 2x_2 &= 0 \\ 2x_1 - x_2 &= 0 \end{aligned}\right\} \rightarrow x_2 = 2x_1$$

$x_1 = 1 \rightarrow x_2 = 2 \rightarrow X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ eigen vector.

$\lambda_2 = 2 , \ X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{aligned} x_1 - 2x_2 &= 0 \\ 2x_1 - 4x_2 &= 0 \end{aligned}\right\} \rightarrow x_1 = 2x_2$$

$x_2 = 1 \rightarrow x_1 = 2 \rightarrow X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ eigen vector.

λ_1 , λ_2 eigen values of A:

Case 1

λ_1 , λ_2 are two distind real numbers $(\lambda_1 \neq \lambda_2)$

$\lambda_1 \rightarrow \xi^{(1)} , \ \lambda_2 \rightarrow \xi^{(2)}$

$$\boxed{x^{(1)} = \xi^{(1)} e^{\lambda_1 t}} \qquad \boxed{x^{(2)} = \xi^{(2)} e^{\lambda_2 t}}$$

$X = C_1 x^{(1)} + C_2 x^{(2)}$ (G.S)

---

e.g: Solve the system:

$$X' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} X \leftarrow A$$

$$|A - \lambda I| = 0 \longrightarrow \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-2-\lambda) + 4 = 0 \longrightarrow -6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - \lambda - 2 = 0 \longrightarrow (\lambda+1)(\lambda-2) = 0 \longrightarrow \boxed{\lambda_1 = -1,\ \lambda_2 = 2}$$ eigen values

$$\lambda_1 = -1, \quad \xi^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{matrix} 4\xi_1 - 2\xi_2 = 0 \\ 2\xi_1 - \xi_2 = 0 \end{matrix}\right] \longrightarrow \xi_2 = 2\xi_1$$

$$\xi_1 = 1 \longrightarrow \xi_2 = 2 \longrightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \longrightarrow X^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

$$\lambda_2 = 2, \quad \xi^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{matrix} \xi_1 - 2\xi_2 = 0 \\ 2\xi_1 - 4\xi_2 = 0 \end{matrix}\right] \longrightarrow \xi_1 = 2\xi_2$$

$$\xi_2 = 1 \longrightarrow \xi_1 = 2 \longrightarrow \xi^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \longrightarrow X^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

The (G.S): $X = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

$$X_1(t) = C_1 e^{-t} + 2C_2 e^{2t}$$
$$X_2(t) = 2C_1 e^{-t} + C_2 e^{2t}$$

e.g: Solve:

$$x' = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}}_{A} x \quad , \quad x(0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)(-2-\lambda) - 4 = 0$$

$$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0 \rightarrow \lambda^2 + \lambda - 6 = 0 \rightarrow (\lambda - 2)(\lambda + 3) = 0$$

$\boxed{\lambda_1 = 2, \lambda_2 = -3}$ eigen values.

$$\lambda_1 = 2 \,,\, \xi^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow -\xi_1 + \xi_2 = 0 \\ \rightarrow 4\xi_1 - 4\xi_2 = 0 \end{matrix} \Big] \rightarrow \xi_1 = \xi_2$$

$$\therefore \xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$\lambda_2 = -3 \,,\, \xi^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow 4\xi_1 + \xi_2 = 0 \\ \rightarrow 4\xi_1 + \xi_2 = 0 \end{matrix} \Big] \rightarrow \xi_2 = -4\xi_1$$

$$\xi_1 = 1 \rightarrow \xi_2 = -4 \rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \longrightarrow x^{(2)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

$$\therefore x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} \quad (G.S)$$

$$x(0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \begin{matrix} \longrightarrow c_1 + c_2 = 4 \\ \longrightarrow c_1 - 4c_2 = -2 \end{matrix}$$

$$5c_2 = 6 \rightarrow \boxed{c_2 = \frac{6}{5}} \longrightarrow \boxed{c_1 = \frac{14}{5}}$$

$$\therefore x = \frac{14}{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \frac{6}{5} \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} \quad (P.S)$$

$\boxed{\text{Case 2}}$

$\lambda_{1,2}$ are complex conjugates:

$\lambda_1 \rightarrow \xi^{(1)} \rightarrow X^{(1)} = u(t) + iv(t)$

$\lambda_2 \rightarrow X^{(2)} = u - iv$

$* \; e^{i\theta} = \cos\theta + i\sin\theta \; *$

$\boxed{X = C_1 u + C_2 v}$ (G.S)

e.g: Solve:

$$X' = \underbrace{\begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}}_{A} X$$

$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 3-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = 0 \rightarrow (3-\lambda)(-1-\lambda) + 8 = 0$

$\rightarrow -3 - 3\lambda + \lambda + \lambda^2 + 8 = 0$

$\lambda^2 - 2\lambda + 5 = 0 \rightarrow \lambda = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm 4i}{2}$

$\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$

$\lambda_1 = 1 + 2i$, $\xi^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

$\begin{pmatrix} 2-2i & -4 \\ 2 & -2-2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow (2-2i)\xi_1 - 4\xi_2 = 0$

$\rightarrow 2\xi_1 - (2+2i)\xi_2 = 0$

$\rightarrow \xi_1 = (1+i)\xi_2$

$\xi_2 = 1 \rightarrow \xi_1 = 1 + i \rightarrow \xi^{(1)} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \rightarrow X^{(1)} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^{t+2ti} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^{t} \cdot e^{2ti}$

$\rightarrow X^{(1)} = \begin{pmatrix} (1+i)\, e^{2ti} \\ e^{2ti} \end{pmatrix} e^{t} = \begin{pmatrix} (1+i)(\cos 2t + i\sin 2t) \\ \cos 2t + i \sin 2t \end{pmatrix} e^{t} = \begin{pmatrix} \cos 2t - \sin 2t + i(\cos 2t + \sin 2t) \\ \cos 2t + i\sin 2t \end{pmatrix} e^{t}$

$\rightarrow X^{(1)} = \underbrace{\begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} e^{t}}_{u} + i \underbrace{\begin{pmatrix} \cos 2t + \sin 2t \\ \sin 2t \end{pmatrix} e^{t}}_{v}$

$\therefore X = C_1 \begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} e^{t} + C_2 \begin{pmatrix} \cos 2t + \sin 2t \\ \sin 2t \end{pmatrix} e^{t}$ (G.S)

Case 3

$\lambda_1 = \lambda_2 \in \mathbb{R}$ (Repeated eigen values)

$\lambda_1 \to \xi^{(1)} \to X^{(1)} = \xi^{(1)} e^{\lambda_1 t}$

$X^{(2)} = \xi^{(1)} t e^{\lambda_1 t} + \eta e^{\lambda_2 t}$, η satisfies $(A - \lambda I)\eta = \xi^{(1)}$

$\therefore X = C_1 X^{(1)} + C_2 X^{(2)}$ (G.S)

e.g: Solve:

$$X' = \underbrace{\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}}_{A} X$$

$|A - \lambda I| = 0 \longrightarrow \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0 \longrightarrow (3-\lambda)(-1-\lambda) + 4 = 0$

$\longrightarrow -3 - 3\lambda + \lambda + \lambda^2 + 4$

$\lambda^2 - 2\lambda + 1 = 0 \longrightarrow (\lambda - 1)(\lambda - 1) = 0 \longrightarrow \boxed{\lambda_1 = \lambda_2 = 1}$ eigen values

$\lambda_1 = 1$, $\xi^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{matrix} \longrightarrow 2\xi_1 - 4\xi_2 = 0 \\ \longrightarrow \xi_1 - 2\xi_2 = 0 \end{matrix} \Bigg] \longrightarrow \xi_1 = 2\xi_2$

$\therefore \xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \longrightarrow X^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$

let $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{matrix} \longrightarrow 2\eta_1 - 4\eta_2 = 2 \\ \longrightarrow \eta_1 - 2\eta_2 = 1 \end{matrix} \Bigg] \longrightarrow \eta_1 = 1 + 2\eta_2$

$\eta_2 = K \longrightarrow \eta_1 = 1 + 2K$

$\eta = \begin{pmatrix} 1+2K \\ K \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \boxed{K \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$ → عديم الفايده

$\therefore X^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^{t} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t}$

$\therefore X = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + C_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^{t} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \right]$ (G.S)

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Ex: Solve:

$\boxed{\text{I}}$ $x' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} x$

$\boxed{2}$ $X' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} X$

Chapter 4

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* Laplace Transformations *
(transforms)

$$\mathcal{L}: \quad \begin{array}{c} t\text{-space} \\ f(t) \\ g(t) \\ h(t) \\ \vdots \end{array} \quad \begin{array}{c} \\ \xrightarrow{\ \mathcal{L}\ } \\ \xrightarrow{\ \mathcal{L}\ } \\ \xleftarrow[\ \mathcal{L}^{-1}\ ]{} \\ \\ \end{array} \quad \begin{array}{c} s\text{-space} \\ F(s) \\ G(s) \\ H(s) \\ \vdots \end{array}$$

Def:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s)$$

Theorem:

① $\mathcal{L}\{f(t) \mp g(t)\} = \mathcal{L}\{f(t)\} \mp \mathcal{L}\{g(t)\}$

② $\mathcal{L}\{c f(t)\} = c\,\mathcal{L}\{f(t)\}$

e.g: by deffinition find:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st}\, dt = \lim_{L\to\infty} \int_0^L e^{(a-s)t}\, dt = \lim_{L\to\infty} \left.\frac{e^{(a-s)t}}{(a-s)}\right]_0^L$$

$$\mathcal{L}\{e^{at}\} = \lim_{L\to\infty} \left[\frac{e^{(a-s)L}}{(a-s)} - \frac{1}{(a-s)}\right] = 0 - \frac{1}{(a-s)} \quad , s > a$$

$$\boxed{\begin{array}{c} e^{\infty} = \infty \\ e^{-\infty} = 0 \end{array}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} = F(s) \quad , \quad s > a$$

مجاله

Ex: $\cosh(at) = \frac{1}{2}e^{at} + \frac{1}{2}e^{-at}$. Find $\mathcal{L}\{\cosh(at)\}$.

Ex: $\sinh(at) = \frac{1}{2}e^{at} - \frac{1}{2}e^{-at}$. Find $\mathcal{L}\{\sinh(at)\}$.

$$\cos(at) + i\sin(at) = e^{i(at)}$$

$$\mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\} = \mathcal{L}\{e^{i(at)}\}$$

$$\mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\} = \frac{1}{S-ia} \cdot \frac{S+ia}{S+ia}$$

$$\mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\} = \frac{S+ia}{S^2+a^2}$$

$$\mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\} = \frac{S}{S^2+a^2} + i\frac{a}{S^2+a^2}$$

$$\mathcal{L}\{\cos(at) = \frac{S}{S^2+a^2}$$

$$\mathcal{L}\{\sin(at) = \frac{a}{S^2+a^2}$$

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$$

$$\vdots$$

$$\mathcal{L}\{t^n\} = \frac{n!}{S^{n+1}}$$

Chapter 4

Day 27 Month 4 Year (2009)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
c	$\frac{c}{s}$
$t^n \quad , \; n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$

$$\mathcal{L}\{\cosh(at)\} = \int_0^\infty \left(\frac{1}{2}e^{at} + \frac{1}{2}e^{-at}\right)e^{-st}\,dt = \int_0^\infty \frac{1}{2}\left(e^{at}\cdot e^{-st} + e^{-at}\cdot e^{-st}\right)dt$$

$$\mathcal{L}\{\cosh(at)\} = \frac{1}{2}\int_0^\infty e^{t(a-s)} + e^{t(-a-s)}\,dt = \frac{1}{2}\left(\frac{e^{t(a-s)}}{a-s} + \frac{e^{t(-a-s)}}{-a-s}\right)\Bigg]_0^\infty$$

$$\mathcal{L}\{\cosh(at)\} = \frac{1}{2}\left[0+0-\frac{1}{a-s} - \frac{1}{-a-s}\right] = \frac{+\cancel{a}+s-\cancel{a}+s}{2(a-s)(-a-s)} = \boxed{\frac{s}{s^2-a^2}}$$

$$\mathcal{L}\{\sinh(at)\} = \int_0^\infty \left(\frac{1}{2}e^{at} - \frac{1}{2}e^{-at}\right)e^{-st}\,dt = \int_0^\infty \frac{1}{2}\left(e^{at}\cdot e^{-st} - e^{-at}\cdot e^{-st}\right)dt$$

$$\mathcal{L}\{\sinh(at)\} = \frac{1}{2}\int_0^\infty e^{t(a-s)} - e^{t(-a-s)}\,dt = \frac{1}{2}\left(\frac{e^{t(a-s)}}{a-s} - \frac{e^{t(-a-s)}}{-a-s}\right)\Bigg]_0^\infty$$

$$\mathcal{L}\{\sinh(at)\} = \frac{1}{2}\left[0-0-\frac{1}{a-s} + \frac{1}{-a-s}\right] = \frac{+a+\cancel{s}+a-\cancel{s}}{2(a-s)(-a-s)} = \boxed{\frac{a}{s^2-a^2}}$$

e.g: Find the Laplace transform:

① $\mathcal{L}\{t^3+6t^2-t+5e^{2t}+\sinh 4t\}$

$$=\frac{3!}{S^4}+6\frac{2!}{S^3}-\frac{1}{S}+5\left(\frac{1}{S-2}\right)+\frac{4}{S^2-16}$$

$$=\frac{6}{S^4}+\frac{12}{S^3}-\frac{1}{S}+\frac{5}{S-2}+\frac{4}{S^2-16}$$

② $\mathcal{L}\{\sin^2 3t\}=\frac{1}{2}\mathcal{L}\{1-\cos 6t\}=\frac{1}{2}\left[\frac{1}{S}-\frac{S}{S^2+36}\right]$

③ $\mathcal{L}\left\{\frac{5}{e^{3t}}+2\right\}=\mathcal{L}\{5e^{-3t}+2\}=5\left(\frac{1}{S+3}\right)+\frac{2}{S}=\frac{5}{S+3}+\frac{2}{S}$

e.g: Find the inverse Laplace transform:

① $\mathcal{L}^{-1}\left\{\frac{7}{S-3}\right\}=7\mathcal{L}^{-1}\left\{\frac{1}{S-3}\right\}=7e^{3t}$

② $\mathcal{L}^{-1}\left\{\frac{S+4}{S^2+25}\right\}=\mathcal{L}^{-1}\left\{\frac{S}{S^2+25}\right\}+\frac{4}{5}\mathcal{L}^{-1}\left\{\frac{5}{S^2+25}\right\}$

$$=\cos 5t+\frac{4}{5}\sin 5t$$

③ $\mathcal{L}^{-1}\left\{\frac{3S+1}{S^2-S-2}\right\}$

$\frac{3S+1}{S^2-S-2}=\frac{A}{S-2}+\frac{B}{S+1}\longrightarrow 3S+1=A(S+1)+B(S-2)$

$S=+2\rightarrow A=\frac{7}{3}$, $S=-1\rightarrow B=\frac{2}{3}$

$\mathcal{L}^{-1}\left\{\frac{3S+1}{S^2-S-2}\right\}=\frac{7}{3}\mathcal{L}^{-1}\left\{\frac{1}{S-2}\right\}+\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{S+1}\right\}$

$$=\frac{7}{3}e^{2t}+\frac{2}{3}e^{-t}$$

④ $\mathcal{L}^{-1}\left\{\frac{2S+1}{S^4+13S^2+36}\right\}$

$$\frac{2S+1}{(S^2+4)(S^2+9)} = \frac{AS+B}{S^2+4} + \frac{CS+D}{S^2+9}$$

$$2S+1 = (AS+B)(S^2+9) + (CS+D)(S^2+4)$$

$S=2i: 4i+1 = (2Ai+B)(5) \rightarrow 4i+1 = 10Ai+5B \rightarrow \boxed{A=\frac{2}{5}}, \boxed{B=\frac{1}{5}}$

$S=3i: 6i+1 = (3Ci+D)(-5) \rightarrow 6i+1 = -15Ci-5D \rightarrow \boxed{C=\frac{-2}{5}}, \boxed{D=\frac{-1}{5}}$

$$\mathcal{L}^{-1}\left\{\frac{2S+1}{S^4+13S^2+36}\right\} = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2*5}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{1}{3*5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2S+1}{S^4+13S^2+36}\right\} = \frac{2}{5}\cos(2t) + \frac{1}{10}\sin(2t) - \frac{2}{5}\cos(3t) - \frac{1}{15}\sin(3t)$$

⑤ $\mathcal{L}^{-1}\left\{\frac{10}{S^6}\right\} = \frac{10}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{S^6}\right\} = \frac{10}{120}t^5 = \frac{1}{12}t^5$

Theorem: S- Shifting theorem

1^{st} S-shifting theorem:

If $\mathcal{L}\{f(t)\} = F(S)$ then: $\mathcal{L}\{f(t)\,e^{at}\} = F(S-a)$

$$\mathcal{L}^{-1}\{F(S-a)\} = e^{at}f(t)$$

e.g: Find the Laplace transform:

① $\mathcal{L}\{t^4 e^{7t}\} = \frac{4!}{(S-7)^5} = \frac{24}{(S-7)^5}$

② $\mathcal{L}\{e^{3t}\cos 4t\} = \dfrac{(S-3)}{(S-3)^2+16}$

③ $\mathcal{L}\{\cosh 2t \cos t\} = \frac{1}{2}\mathcal{L}\{(e^{2t}+e^{-2t})\cos t\}$

$= \frac{1}{2}\mathcal{L}\{e^{2t}\cos t\} + \frac{1}{2}\mathcal{L}\{e^{-2t}\cos t\}$

$= \frac{1}{2}\left(\dfrac{S-2}{(S-2)^2+1}\right) + \frac{1}{2}\left(\dfrac{S+2}{(S+2)^2+1}\right)$

e.g: Find the inverse Laplace transform:

① $\mathcal{L}^{-1}\left\{\dfrac{5}{(S+3)^7}\right\} = \dfrac{5}{6!}\mathcal{L}\left\{\dfrac{6!}{(S-3)^7}\right\} = \dfrac{5}{720} t^6 e^{3t}$

② $\mathcal{L}^{-1}\left\{\dfrac{(S-2)}{(S-2)^2-9}\right\} = e^{2t}\cosh 3t$

③ $\mathcal{L}^{-1}\left\{\dfrac{5S+1}{(S+2)^2+3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{5(S+2-2)+1}{(S+2)^2+3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{5(S+2)-9}{(S+2)^2+3}\right\}$

$= 5\mathcal{L}^{-1}\left\{\dfrac{S+2}{(S+2)^2+3}\right\} - \dfrac{9}{\sqrt{3}}\mathcal{L}^{-1}\left\{\dfrac{\sqrt{3}}{(S+2)^2+3}\right\}$

$= 5e^{-2t}\cos(\sqrt{3}\,t) - 3\sqrt{3}\,e^{-2t}\sin(\sqrt{3}\,t)$

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$

$\rightarrow F'(s) = \int_0^\infty -t f(t) e^{-st}\,dt \rightarrow -F'(s) = \int_0^\infty t f(t) e^{-st}\,dt$

$\therefore \mathcal{L}\{t f(t)\} = -F'(s)$

$+F''(s) = +\int_0^\infty t^2 f(t) e^{-st}\,dt \rightarrow -F'''(s) = \int_0^\infty t^3 f(t) e^{-st}\,dt$

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(S)}$$

Theorem:

If $\mathcal{L}\{f(t)\} = F(s)$ then:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau$$

e.g: Find:

① $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s-2}\right)\right\} = \int_0^t e^{2\tau}\,d\tau = \frac{1}{2}e^{2\tau}\Big]_0^t = \frac{1}{2}e^{2t} - \frac{1}{2}$

② $\mathcal{L}^{-1}\left\{\frac{1}{s^3-2s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s^2-2s}\right)\right\} = \int_0^t \frac{1}{2}(e^{2\tau}-1)\,d\tau$

$= \frac{1}{2}\left[\frac{1}{2}e^{2\tau} - \tau\right]_0^t = \frac{1}{2}\left[\left(\frac{1}{2}e^{2t} - t\right) - \frac{1}{2}\right] = \frac{1}{4}e^{2t} - \frac{1}{2}t - \frac{1}{4}$

Ex: Find:

① $\mathcal{L}^{-1}\left\{\frac{1}{s^4+9s^2}\right\}$

② $\mathcal{L}^{-1}\{\ln(s-4)\}$

③ $\mathcal{L}^{-1}\left\{\cos^{-1}\left(\frac{s}{\pi}\right)\right\}$

$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y'(t)\} = S\,Y(S) - y(0)$

$\mathcal{L}\{y''(t)\} = S^2 Y(S) - S y(0) - y'(0)$

$\mathcal{L}\{y'''(t)\} = S^3 Y(S) - S^2 y(0) - S\,y'(0) - y''(0)$

$$\boxed{\mathcal{L}\{y^{(n)}(t)\} = S^n Y(S) - S^{n-1} y(0) - S^{n-2} y'(0) \ldots\ldots y^{(n-1)}(0)}$$

e.g: Solve:

① $y' - y = 2 \quad , \quad y(0) = 0$

$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{2\}$

$S\,Y(S) - \cancel{y(0)}^{0} - Y(S) = \frac{2}{S}$

$Y(S)(S-1) = \frac{2}{S} \longrightarrow Y(S) = \frac{2}{S(S-1)}$

$\mathcal{L}^{-1}\left\{\frac{2}{S(S-1)}\right\} = 2\,\mathcal{L}^{-1}\left[\frac{1}{S}\left(\frac{1}{S-1}\right)\right] = 2\int_0^t e^{\tau}\, d\tau = 2e^{\tau}\Big]_0^t$

$y(t) = 2e^t - 2$

② $y^{iv} - 4y = 0 \quad , y(0) = 1 , y'(0) = 0 , y''(0) = -2 , y'''(0) = 0$

$\mathcal{L}\{y^{iv}\} - 4\,\mathcal{L}\{y\} = \mathcal{L}\{0\} \longrightarrow S^4 Y(S) - S^3 y(0) - S^2 \cancel{y'(0)}^{0} - S y''(0) - \cancel{y'''(0)}^{0} - 4Y(S) = 0$

$S^4 Y(S) - S^3 + 2S - 4Y(S) = 0 \rightarrow Y(S)(S^4 - 4) = S^3 - 2S \rightarrow Y(S) = \frac{S^3 - 2S}{S^4 - 4}$

$Y(S) = \frac{S\cancel{(S^2-2)}}{\cancel{(S^2-2)}(S^2+2)} = \frac{S}{S^2+2} \qquad \therefore y(t) = \mathcal{L}^{-1}\left\{\frac{S}{S^2+2}\right\}$

$\therefore y(t) = \cos\sqrt{2}\,t$

③ $y'' - 2y' + y = 4e^{-t}$, $y(0) = 2$, $y'(0) = -1$

$$S^2 Y(s) - Sy(0) - y'(0) - 2SY(S) + 2y(0) + Y(S) = \frac{4}{S+1}$$
$$S^2 Y(S) - 2S + 1 - 2SY(S) + 4 + Y(S) = \frac{4}{S+1}$$
$$Y(S)(S^2 - 2S + 1) = \left(\frac{4}{S+1}\right) + (2S - 5)$$
$$Y(S) = \frac{4}{(S+1)(S-1)^2} + \frac{2S-5}{(S-1)^2} = \frac{4 + (2S-5)(S+1)}{(S+1)(S-1)^2}$$
$$Y(S) = \frac{2S^2 - 3S - 1}{(S-1)^2(S+1)} \longrightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{2S^2 - 3S + 1}{(S-1)^2(S+1)}\right\}$$
$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{S-1} + \frac{3}{(S-1)^2}\right\} = 2e^{t} + 3te^{t}$$

Ex: ④ $y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = -1$

$$S^2 Y(S) - Sy(0) - y'(0) - SY(S) + y(0) - 6Y(S) = 0$$
$$S^2 Y(S) - S + 1 - SY(S) + 1 - 6Y(S) = 0$$
$$Y(S)(S^2 - S - 6) = S - 2 \longrightarrow Y(S) = \frac{S-2}{(S-3)(S+2)}$$
$$y(t) = \mathcal{L}^{-1}\left\{\frac{S-2}{(S-3)(S+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{5}}{S-3} + \frac{\frac{4}{5}}{S+2}\right\}$$
$$y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

Ex: ⑤ $y'' + y = t$, $y(0) = 0$, $y'(0) = 2$

$$S^2 Y(S) - \cancel{Sy(0)} - y'(0) + Y(S) = \frac{1}{S^2}$$
$$Y(S)(S^2 + 1) = \frac{1}{S^2} + 2 \longrightarrow Y(S) = \frac{1 + 2S^2}{S^2(S^2+1)}$$
$$y(t) = \mathcal{L}^{-1}\left\{\frac{1 + 2S^2}{S^2(S^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{S^2} + \frac{1}{S^2+1}\right\}$$
$$y(t) = t + \sin t$$

Ex: 6 $y'' - 2y' + 2y = 0 \quad , y(0) = 2 , y'(0) = -1$

$$S^2 Y(s) - S y(0) - y'(0) - 2 S Y(s) + 2 y(0) + 2 Y(s) = 0$$
$$S^2 Y(s) - 2 S + 1 - 2 S Y(s) + 4 + 2 Y(s) = 0$$
$$Y(s) (S^2 - 2S + 2) = 2S - 5$$
$$Y(s) = \frac{2S - 5}{S^2 - 2S + 2}$$

$+ t y'$

$$\mathcal{L}\{t y'\} = \frac{-d}{ds}\left[s Y(s) - y(0)\right]$$

$$= -\left[-s Y'(s) + Y(s)\right]$$

Chapter 5

Day 4 Month 5 Year (2009)

Fourier Series and Fourier Integral:

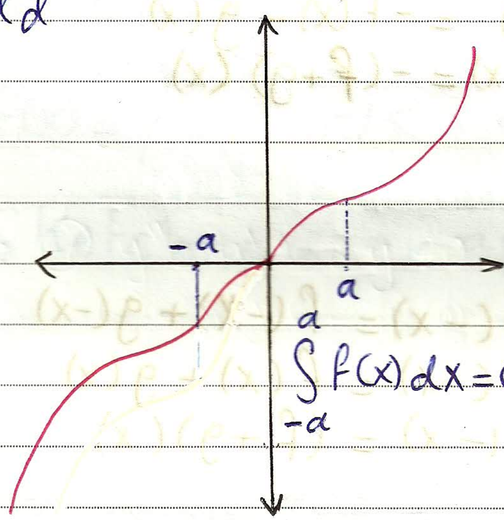
Introduction:

Def: A function $f(x)$ is called

① an odd function if $f(-x) = -f(x) \quad \forall x \in D_f$

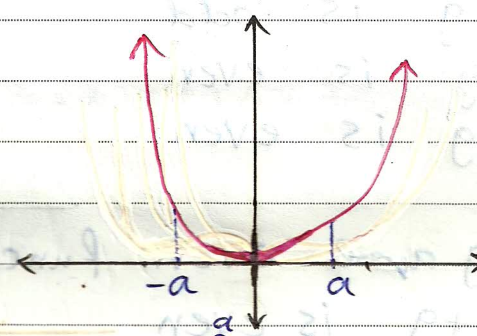
② an even function if $f(-x) = f(x) \quad \forall x \in D_f$

odd



$$\int_{-a}^{a} f(x)\,dx = 0$$

even



$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

e.g: which of the following functions is odd, even or neither?

① $f(x) = x^3 - 2x$

$f(-x) = -x^3 + 2x = -[x^3 - 2x] = -f(x) \quad \therefore f(x)$ is an odd.

② $f(x) = x^4 + 1$

$f(-x) = x^4 + 1 = f(x) \quad \therefore f(x)$ is even.

③ $f(x) = x^2 + x^5$

$f(-x) = x^2 - x^5 \neq f(x) \rightarrow -[x^2 + x^5] \neq -f(x)$

$\therefore f(x)$ is neither odd nor even.

④ $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x = f(x) \quad \therefore f(x)$ is even.

⑤ $f(x) = \sin(4x)$

$f(-x) = \sin(-4x) = -\sin(4x) = -f(x)$ $\therefore f(x)$ is an odd.

⑥ $f(x) = |x|$

$f(-x) = |-x| = |x| = f(x)$ $\therefore f(x)$ is **even**.

f, g are odd functions:

1. f+g is odd
2. f−g is odd
3. fg is even
4. f/g is even

$$\begin{cases} (f+g)(-x) = f(-x) + g(-x) \\ (f+g)(-x) = -f(x) - g(x) \\ (f+g)(-x) = -(f+g)(x) \end{cases}$$

f, g are even functions:

1. f+g is even
2. f−g is even
3. fg is even
4. f/g is even

$$\begin{cases} (f+g)(-x) = f(-x) + g(-x) \\ (f+g)(-x) = f(x) + g(x) \\ (f+g)(-x) = (f+g)(x) \end{cases}$$

If f is odd and g is even then:

1. f+g is neither
2. f−g is neither
3. fg is odd
4. f/g is odd

Def: A function f is called periodic function:
if there exists a positive real number M such that $f(x+M) = f(x)$ $\forall x$. and the smallest such M is called the (principal) period.

Chapter 5

Day 4 Month 5 Year (2009)

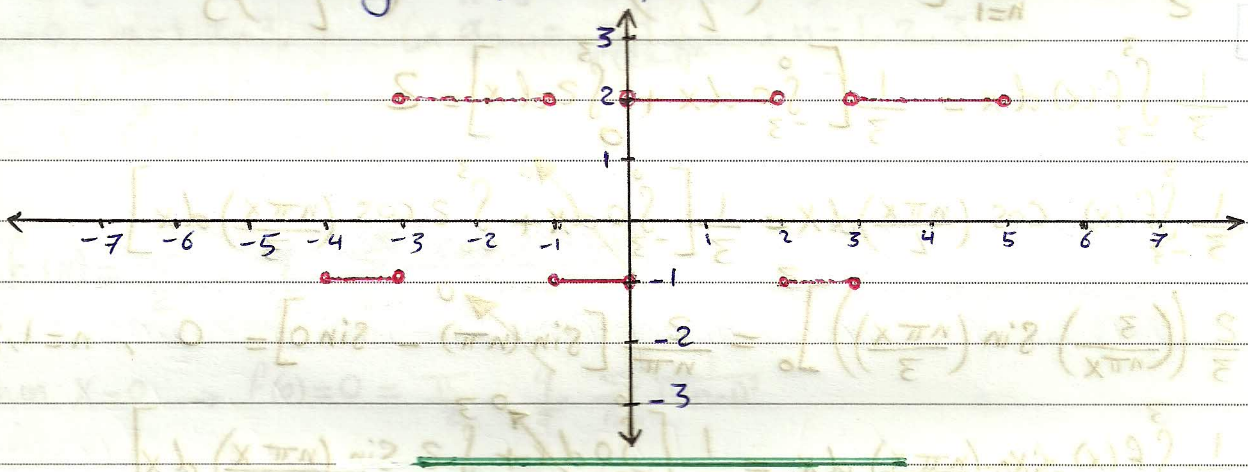
Sin (nx) is aperiodic function its period $\frac{2\pi}{n}$
Cos (nx) is aperiodic function its period $\frac{2\pi}{n}$
tan (nx) is aperiodic function its period $\frac{\pi}{n}$
Cot (nx) is aperiodic function its period $\frac{\pi}{n}$
sec (nx) is aperiodic function its period $\frac{2\pi}{n}$
csc (nx) is aperiodic function its period $\frac{2\pi}{n}$

---

e.g: Given that:

$$f(x) = \begin{cases} 2 & , 0 < x < 2 \\ -1 & , -1 < x < 0 \end{cases} \quad , f(x) = f(x+3) \; \forall \, x$$

Sketch the graph of f(x).



Theorem: (Forier theorem)

If f is pwc on $(-L, L) \rightarrow f(x) = f(x - 2L) \; \forall \, x$
then the fourier series corresponding to f (x) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\} \longrightarrow \otimes$$

where: $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ } Fourier coefficients

The series in ⊛ converges to $\frac{f(x_0+0)+f(x_0-0)}{2}$ when $x = x_0$

$$f(x_0+0) = \lim_{x \to x_0^+} f(x)$$

$$f(x_0-0) = \lim_{x \to x_0^-} f(x)$$

e.g: If $f(x) = \begin{cases} 2 & , 0 < x < 3 \\ 0 & , -3 < x < 0 \end{cases}$, $f(x) = f(x+6)\ \forall x$

Find fourier series corresponding to $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

$\boxed{L=3}$

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\left[\int_{-3}^{0} 0\,dx + \int_{0}^{3} 2\,dx\right] = 2$$

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{3}\left[\int_{-3}^{0} 0\,dx + \int_{0}^{3} 2\cos\left(\frac{n\pi x}{3}\right)dx\right]$$

$$a_n = \frac{2}{3}\left(\left(\frac{3}{n\pi x}\right)\sin\left(\frac{n\pi x}{3}\right)\right)\Bigg]_0^3 = \frac{2}{n\pi}\left[\sin(n\pi) - \sin 0\right] = 0 \quad , n = 1,2,3,\ldots$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{1}{3}\left[\int_{-3}^{0} 0\,dx + \int_{0}^{3} 2\sin\left(\frac{n\pi x}{3}\right)dx\right]$$

$$b_n = \frac{2}{3}\left(\left(\frac{-3}{n\pi x}\right)\cos\left(\frac{n\pi x}{3}\right)\right)\Bigg]_0^3 = \frac{-2}{n\pi}\left[\cos(n\pi) - \cos 0\right] = \frac{-2}{n\pi}\left[(-1)^n - 1\right] \quad , n=1,2,3,\ldots$$

$b_{2n} = 0 \quad , n = 1,2,3,\ldots$, $b_{2n-1} = \frac{-2}{(2n-1)\pi}\left[-1-1\right] \rightarrow b_{2n-1} = \frac{-4}{\pi(2n-1)}$, $n = 1,2,3,\ldots$

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\left\{0 + \frac{4}{\pi(2n-1)}\sin\left(\frac{(2n-1)\pi x}{3}\right)\right\}$$

$$\therefore f(x) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{(2n-1)\pi x}{3}\right)}{2n-1}$$

the series converges to $\frac{2+0}{2} = 1$

e.g: Given that:

$f(x) = |x| \quad , \quad x \in (-\pi, \pi)$ $[-\pi, \pi]$

Find the fourier series corresponding to $f(x)$. and hence, find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$\boxed{L = \pi}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{2}{\pi} \int_0^{\pi} x \, dx = \frac{2}{\pi} \cdot \frac{x^2}{2} \Big]_0^{\pi} = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) \, dx = \frac{2}{\pi} \left[\frac{1}{n} x \sin(nx) + \frac{1}{n^2} \cos(nx) \right] \Big]_0^{\pi}$$

$$a_n = \frac{2}{\pi} \left\{ 0 + \frac{1}{n^2} \cos n\pi - \frac{1}{n^2} \right\} = \frac{2}{\pi n^2} \left((-1)^n - 1 \right) \quad , n = 1, 2, 3, \dots$$

$$a_{2n} = 0, \quad n = 1, 2, 3, \dots \quad , \quad a_{2n-1} = \frac{-4}{\pi (2n-1)^2} \quad , n = 1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) \, dx = 0 \qquad \left(\text{odd} * \text{even} = \text{odd} \rightarrow \int_{-a}^{a} f(x) dx = 0 \right)$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

When $x = 0 \rightarrow f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$$\rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Half range Fourier Series:

$$(\dots\dots|\underline{\quad\quad}) \quad 0 \quad L$$

Cosine Fourier series for the half range $(0, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where:

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \quad , \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Sine Fourier series for the half rang $(0, L)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where: $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Ex: Find the Fourier series corresponding to the function

$$f(x) = \begin{cases} -1 & , -2 \leq x < 0 \\ 1 & , 0 \leq x < 2 \end{cases} \quad , f(x) = f(x+4) \quad \forall x$$

Ex: Find the Fourier Series corresponding to the function

$$f(x) = \begin{cases} x & , -\pi \leq x < 0 \\ 0 & , 0 \leq x < \pi \end{cases} , \quad f(x) = f(x+2\pi) \ \forall x$$

Ex: Find the Fourier Cosine Series for: $f(x) = \sin x$, $x \in (0, \pi)$

Partial Defferential Equations:

① Heat Equation

$u_t(x,t) = a^2 u_{xx}(x,t)$

② Wave Equation

$u_{tt}(x,t) = a^2 u_{xx}(x,t)$

③ Laplace Equation

$u_{xx}(x,y) + u_{yy}(x,y) = 0$

$\nabla^2 u = 0$

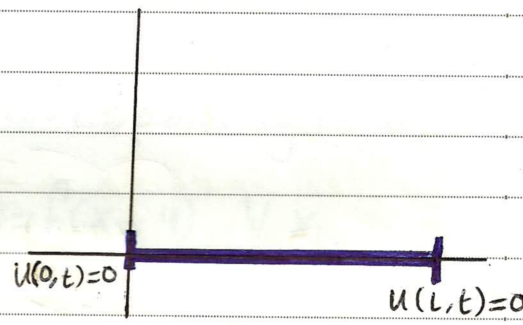
Seperation of variables method:

e.g: Solve:

$u_t(x,t) = a^2 u_{xx}(x,t)$

homogeneous conditions ← $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$, $t > 0$

non-homo condition ← $u(x,0) = f(x)$



let $u(x,t) = X(x)T(t)$

$u_t = X T'$

$u_{xx} = X''(x) T$

Sub. in the D.E: $X T' = a^2 X'' T$

$\div a^2 X T$: $\frac{T'}{a^2 T} = \frac{X''}{X} =$ constant $\forall x \ \forall t$

$\frac{T'}{a^2 T} = \frac{X''}{X} = -\sigma$ ← constant

$X'' + \sigma X = 0$, $X(0) = 0$, $X(L) = 0$

$\boxed{\frac{T'}{T} = -\sigma a^2}$

$u(0,t) = 0$

$X(0) T(t) = 0 \ \forall t$

$\boxed{X(0) = 0}$

$u(L,t) = 0$

$X(L) T(t) = 0$

$\boxed{X(L) = 0}$

$\boxed{\text{Case 1} \quad \sigma < 0}$: $\sigma = -\lambda^2$, $\lambda \neq 0$

$X'' - \lambda^2 X = 0$, $X(0) = 0$, $X(L) = 0$

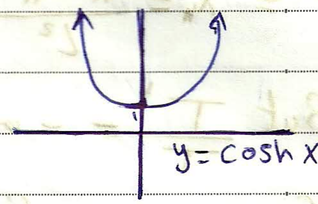
$m^2 - \lambda^2 = 0 \longrightarrow m = \pm\lambda$

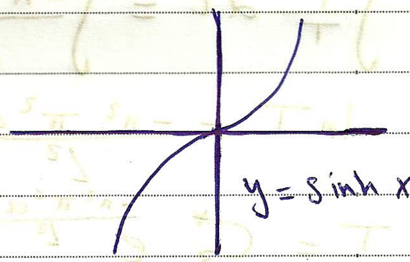
$X = C_1 \cosh\lambda x + C_2 \sinh\lambda x$

$X(0) = 0 \longrightarrow C_1 = 0$

$X(L) = 0 \longrightarrow 0 + C_2 \sinh\lambda L = 0 \rightarrow C_2 = 0$

$\therefore X = 0 \longrightarrow u = 0$ (trivial Solution)





$\boxed{\text{Case 2} \quad \sigma = 0}$: $\sigma = 0$

$X'' = 0$, $X(0) = 0$, $X(L) = 0$

$X = C_1 + C_2 x$

$X(0) = 0 \longrightarrow C_1 = 0$

$X(L) = 0 \longrightarrow 0 + C_2 L = 0 \longrightarrow C_2 = 0$

$\therefore X = 0 \longrightarrow u = 0$ (trivial Solution)

$\boxed{\text{Case 3} \quad \sigma > 0}$: $\sigma = \lambda^2$, $\lambda \neq 0$

$X'' + \lambda^2 X = 0$, $X(0) = 0$, $X(L) = 0$

$m_{1,2} = \pm\lambda i$

$X = C_1 \cos\lambda x + C_2 \sin\lambda x$

$X(0) = 0 \longrightarrow C_1 + 0 = 0 \longrightarrow C_1 = 0$

$X(L) = 0 \longrightarrow 0 + C_2 \sin\lambda L = 0$

$C_2 \neq 0 \longrightarrow \sin\lambda L = 0 \longrightarrow \lambda L = n\pi$, $n = 1, 2, 3, \ldots$

$\boxed{\lambda_n = \frac{n\pi}{L} \quad , n = 1, 2, 3, \ldots}$ → eigen values

$\boxed{X_n = \sin\left(\frac{n\pi}{L} x\right) \quad , n = 1, 2, 3, \ldots}$ → eigen (vector / function)

$$\sigma = \lambda_n^2 = \frac{n^2\pi^2}{L^2}$$

But $\frac{T'}{T} = -\sigma a^2$

$$\int \frac{T'}{T}\, dt = \int \frac{-n^2\pi^2 a^2}{L^2}\, dt$$

$$\ln T = \frac{-n^2\pi^2 a^2}{L^2} t + C_3$$

$$T = C_3^* e^{\frac{-n^2\pi^2 a^2 t}{L^2}}$$

$$T_n = e^{\frac{-n^2\pi^2 a^2 t}{L^2}} \quad , n = 1, 2, 3, \ldots$$

$$u_n(x,t) = e^{\frac{-n^2\pi^2 a^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right) \quad , n = 1, 2, 3, \ldots$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n e^{\frac{-n^2\pi^2 a^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)} \quad (G.S)$$

But:

$$u(x,0) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \quad 0 < x < L$$

$$C_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Chapter 6

e.g: Find the Solution of

$$u_t(x,t) = 100\,u_{xx}(x,t) \qquad t>0,\ 0\leq x\leq 1$$

$$u(0,t) = u(1,t) = 0 \quad \forall t$$

$$u(x,0) = \sin(2\pi x) - 2\sin(5\pi x)$$

$$a = 10,\quad L = 1$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-100\pi^2 n^2 t} \sin(n\pi x)$$

$$u(x,0) = \sin(2\pi x) - 2\sin(5\pi x)$$

$$\sin(2\pi x) - 2\sin(5\pi x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

$$\rightarrow C_2 = 1,\quad C_5 = -2$$

$$0 = C_1 = C_3 = C_4 = C_6 = C_7 = C_8 = \dots$$

$$\boxed{u(x,t) = e^{-400\pi^2 t}\sin(2\pi x) - 2e^{-2500\pi^2 t}\sin(5\pi x)}$$

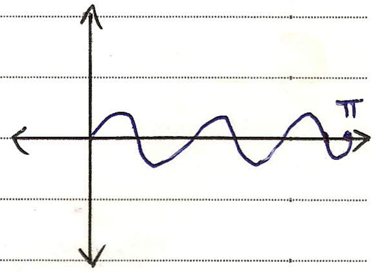
*ملاحظة : الجواب النهائي غالباً series وهنا اقتران محدود.

e.g: Solve:

$$u_{tt}(x,t) = a^2 u_{xx}(x,t),\quad 0<x<\pi,\quad t>0$$

$$u_x(0,t) = u_x(\pi,t) = 0,\quad t>0$$

$$u_t(x,0) = 0,\quad u(x,0) = kx,\quad k:\ \text{Constant}$$



let $u(x,t) = X(x)T(t)$

$$u_{tt} = X T''$$

$$u_{xx} = X'' T$$

Sub. in the D.E: $XT'' = a^2 X'' T$

$$\div a^2 XT:\ \frac{T''}{a^2 T} = \frac{X''}{X} = -\sigma$$

Chapter 6

Day 13 Month 5 Year (2009)

$X'' + \sigma X = 0 \quad , X'(0) = 0 \quad , X'(\pi) = 0$

$T'' + a^2 \sigma T = 0 \quad , T'(0) = 0$

Case 1 $\sigma < 0$: (trivial Solution)

Case 2 $\sigma = 0$:

$$u(x,t) = \frac{K\pi}{2} - \frac{4K}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x\cos(2n-1)at}{(2n-1)^2}$$

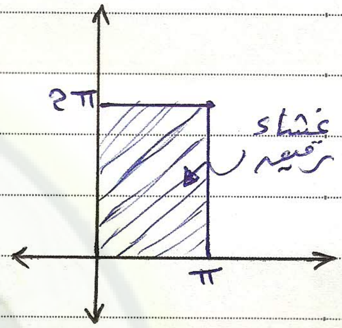
e.g: Solve :

$\nabla^2 u = 0$

$u_{xx} + u_{yy} = 0 \quad , 0 < x < \pi \quad , 0 < y < 2\pi$

$u(0, y) = u(\pi, y) = 0$

$u(x, 0) = 0 \quad , u(x, 2\pi) = 1$



let $u = X(x)\,Y(y)$

$X''Y + XY'' = 0$

$\div XY: \dfrac{X''}{X} + \dfrac{Y''}{Y} = 0$

$\dfrac{X''}{X} = \dfrac{-Y''}{Y} = -\sigma$

$X'' + \sigma X = 0 \quad , X(0) = X(\pi) = 0$

$Y'' - \sigma Y = 0 \quad , Y(0) = 0$

$$U(x,y) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x \, \sin(2n-1)y}{(2n-1)\sinh(4n-2)\pi}$$

* غالباً إذا كانت الشروط على الاقتران فإن case 2 تكون trivial solution
أما إذا كانت الشروط على المشتقة فتعطي حل.

تم بحمد الله وفضله

[لا تنسونا من صالح دعائكم] مهندس متفائل ^_^

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